THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. V, No. 21—March, 1911.

(Whole Series, Vol. XV, No. 21.)

CONTENTS:

FOUNDATIONS OF ARITHMETIC..... *Arthur Bowes Frizell.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

March, 1912.

Entered at the post-office in Lawrence as second-class matter.

3-3310

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. V, No. 21] MARCH, 1911. [WHOLE SERIES VOL. XV, No. 21

THE FOUNDATIONS OF ARITHMETIC.

DISSERTATION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL OF THE UNIVERSITY OF KANSAS.

BY ARTHUR BOWES FRIZELL, OF BOSTON.

This thesis seeks a foundation for arithmetic in the ideas underlying Cantor's formulation of his system of ordinal types.

It proceeds by postulating, but follows D. Hilbert and G. Peano rather than E. V. Huntington. The search is for postulates possessing heuristic and didactic, not merely subsumptive value.

A *motif* is found in the notion of an abstract group, which secures the development step by step of all number systems so far studied without further postulates than those needed for the transfinite ordinals.

As axioms are to be avoided, it is necessary to state carefully the definitions and theorems used even when they are well known to the mathematicians, but in this case the proofs are omitted.

1. *Definition.* A set of symbols $a, b, \ldots$ will be said to form a K-class if we possess a test which enables us to assert in every case either that $a = b$ or that a is not $= b$ subject only to the restrictions that

a) the result of comparison be uniquely determined
b) the statements $b=a$ and $a=b$ shall be interchangeable
c) from $a=b$ and $b=c$ must follow $a=c$
d) $a=a$ if K contains no other symbol equal to a.

2. *Definition.* An infinite set of symbols is one which contains parts that can be put into one correspondence with the whole. A finite set is one that is not infinite.

3. *Definition.* An ordered set is one in which we are always able to say either that a precedes b $(a<b)$ or that a follows b $(a>b)$ subject only to the restrictions that
a) the result of comparison is uniquely determined
b) $a<b$ shall exclude $a=b$ but involve $b>a$
c) if $a<b$ and $b<c$, then $a<c$
d) if $a'=a$ and $b'=b$, then $a'<b'$ or $a'>b'$ according as $a<b$ or $a>b$.

4. *Definition.* An ordered set is said to be well ordered when it has a first element and every subset beginning with the first element has an immediate successor in the given set.

5. *Definition.* A procedure whereby to every a, b of a K-class is assigned a definite symbol $c=a\circ b$ will be called a C-rule or rule of combination provided that by this assignment equals with equals give equals.

6. *Definition.* A K-class is said to possess the fundamental group property with respect to a C-rule when $a\circ b$ also belongs to K.

7. *Definition.* Modulus of a K-class with regard to a C-rule is a symbol u of K such that $a\circ u=a=u\circ a$ for every a in K.

8. *Proposition I.* An ordered set defined by the requirements: **a**) it shall contain a given symbol e; **b**) it shall possess the group property for every combi-

nation $e \circ a$ where a denotes a previously defined member of the set is also well ordered, infinite and forms a K-class with respect to the given rule. *Proof:* By hypothesis the set is to contain $e \circ e = e'$, $e \circ e' = e''$, $e \circ e'' = e'''$, . . . Since it is to be an ordered set no two of its members can be equal $\therefore$ to every element of the whole set can be assigned some one of the subset e', e''', e^v, . . . That is, the set is infinite. And by **b**) every element has an immediate successor. Finally we have a test for equality which satisfies the requirements of § 1.

9. *Scholium.* A set defined as in § 8 contains no modulus. In what follows it will be referred to as an e-set.

10. *Postulates.* We postulate an e-set for a lower rule of combination which shall contain a lower symbol u and e-sets for both this and a higher rule which shall both contain a higher symbol w, the definitions of the rules of combination to be completed in §§ 17, 26, 28, 30.

11. The different sets $K[u\circ]$, $K[w\circ]$, $K[w\square]$ are to be so ordered that the lower shall precede the higher. Thus they form together a well ordered set

$$u,\ u\circ u = u',\ u\circ u' = u'',\ u\circ u'' = u''',\ \ldots$$
$$w,\ w\circ w = wu',\ w\circ wu' = wu'',\ w\circ wu'' = wu''',\ \ldots$$
$$w\square w = w^{u'},\ w\square w^{u'} = w^{u''},\ w\square w^{u''} = w^{u'''},\ \ldots$$

12. *Postulates.* We postulate e-sets for the lower rule which shall contain respectively each of the symbols $w^{u'}$, $w^{u''}$, $w^{u'''}$, . . . in succession. Thus corresponding to every a of the set $K[w\square]$ we shall have an e-set $K[a\circ]: a,\ a\circ a = au',\ a\circ au' = au''$, . . . Now taking *e. g.* $a = w^{u'''}$, by the principle of § 11 every member of $K[w^{u'''}\circ]$ precedes $ww^{u'''} = w^{u^{iv}}$ and therefore falls between $w^{u'''}$ and $w^{u^{iv}}$. That is, all these new e-sets are interpolated between successive

elements of $K[w\square]$. Hence the totality of symbols forms a well ordered set.

13. *Proposition II.* A rule of combination is associative throughout a given K-class if

$$a \circ (b \circ c) = a \circ b \circ c$$

for every a, b, c in K.

14. *Definition.* A K-class is said to form a semigroup with regard to a C-rule associative throughout K if by this rule unequals with equals always give unequals or if from the relation $a' \circ b = a \circ b$, resp. $a \circ b' = a \circ b$ we can always infer $a' = a$ resp. $b' = b$.

15. *Proposition III.* No semigroup contains more than one modulus for its defining C-rule. For if u and u' not $= u$ were both moduli, we should have $a \circ u' = a = a \circ u$ contrary to § 14.

16. *Definition.* An abelian K-class with reference to a C-rule is one for which the rule is without exception commutative.

17. *Proposition IV.* Sufficient conditions of an abelian class are the relations

$$a \circ (b \circ c) = a \circ b \circ c \quad \text{and} \quad b \circ a = a \circ b$$

for every a, b, c in the class.

18. *Proposition V.* A C-rule is distributive over another in a given K-class if the relations

$$(a \circ b)c = ac \circ bc \quad \text{and} \quad a(b \circ c) = ab \circ ac$$

are satisfied by every a, b, c in K.

19. *Proposition VI.* A necessary and sufficient condition of the generating rule of an e-set being associative for the set is the inductive formula of definition.

$$e \circ a \circ b = e \circ (a \circ b).$$

Proof. If the relation of $a \circ (b \circ c) = a \circ b \circ c$ has been established for a certain a and every b, c of the set, then by hypothesis $e \circ a \circ (b \circ c) = e \circ \{a \circ (b \circ c)\} = e \circ \{a \circ b \circ c\} = e \circ (a \circ b) \circ c = e \circ a \circ b \circ c$.

But $e \circ b \circ c = e \circ (b \circ c)$ by definition.
Therefore $e' \circ b \circ c = e' \circ (b \circ c)$ and so on.
Hence the theorem by strict induction.

20. *Corollary.* The rule is also commutative. For if $e \circ a = a \circ e$, then by the associative law $e \circ a \circ e = e \circ (a \circ e) = e \circ (e \circ a)$. But $e \circ e' = e' \circ e$ by § 19, therefore $e \circ e'' = e'' \circ e$, and so on. And if $b \circ a = a \circ b$ for a certain a and every b then $b \circ (e \circ a) = b \circ (a \circ e) = b \circ a \circ e = a \circ b \circ e = e \circ (a \circ b) = e \circ a \circ b$.

Hence the proposition follows by strict induction.

21. *Proposition VII.* The formula of § 19 is also a sufficient condition of the e-set constituting an abelian semigroup for its generating C-rule. *Proof.* If $a \circ b$ belongs to the given e-set, then $e \circ a \circ b = e \circ (a \circ b)$ also belongs to it. Thus the first group property is established. We have proved the associative law and by definition no two numbers of the set are equal. Therefore $a' \circ b$ is not $= a \circ b$ when a' is not $= a$.

22. *Postulates.* Using capital letters M, N, ... to denote elements of the lowest class $\mathrm{K}[u \circ]$, and italic letters a, b, . . . for those of the class $\mathrm{K}[w \square]$ where a shall precede b, we postulate for every $MB = BM$ a set $MB \circ Na$ for every Na, then using letters a, b, ... where a shall precede b to denote the resulting symbols we postulate inductively new sets $\mathrm{b} \circ \mathrm{a}$ where $\mathrm{a} = Na$ and b denotes successively the higher "polynomial" or composite symbols. Every b-set is to be simply ordered and possess the first group property for every $\mathrm{b} \circ \mathrm{a}$.

23. *Proposition VIII.* Every b-set is well ordered, infinite and forms a K-class for the rule denoted by $\circ$. For the b-set consists of the combinations b, $\mathrm{b} \circ \mathrm{a}$, $\mathrm{b} \circ \mathrm{a}\, u'$, $\mathrm{b} \circ \mathrm{a}\, u''$, $\mathrm{b} \circ \mathrm{a}\, u'''$, ... and the reasoning of *Prop. I* holds good.

24. *Proposition IX.* The b-sets belonging to a

given Mb form together an infinite, well ordered K-class as regards the rule $\circ$. For the set of b's:

$$Mb,\ Mb \circ w^{(n)},\ Mb \circ w^{(n \circ u)},\ \ldots .$$

forms a well ordered K-class between consecutive members of which the b-sets are interpolated.

25. *Proposition X.* The totality of the b-sets forms an infinite, well ordered K-class as regards the rule $\circ$. For the set of Mb belonging to a given b is a well ordered K-class and so is the whole set of b's, the latter being simply $K[w \square]$.

26. New combinations of symbols already defined are defined inductively in accordance with the formulas

$$b \circ a = a \circ b \quad \text{and}$$
$$a \circ (b \circ c) = a \circ b \circ c.$$

This secures the group property for the whole set and the reasoning of *VI* and *VII* establishes

Proposition XI. The set of symbols generated according to two rules, a higher and a lower, from a single arbitrary symbol w by aid of the postulate of order, the restricted form of the group property and the above inducted formulas of definition, is a well ordered, infinite set constituting an abelian semigroup with regard to the lower rule.

27. *Scholium.* The higher rule remains unrestricted and has not been defined beyond the set $K[w \square]$, therefore is not yet a C-rule for the K-class of § 26.

28. *Definition.* The higher rule shall be defined inductively throughout $K[w \square]$ by the formula

$$w\,a\,b = w \square a\,b,$$

where a, b are any members of $K[w \square]$ for which ab has already been defined.

29. *Proposition XII.* The class $K[w \square]$ forms an abelian semigroup with regard to its generating rule. Proof by *Propositions VI* and *VII.*

30. *Definition.* The higher rule shall be defined inductively for the remaining symbols of our abelian semigroup on the lower rule according to the formulas

$$(a \circ b)c = ac \circ bc \text{ and } a(b \circ c) = ab \circ ac,$$

where the combinations ab, ac, bc shall have been already defined.

31. *Proposition XIII.* The higher rule is distributive over the lower throughout the class of symbols defined in § 22. *Proof.* First let $a = b = c = l$ where l is any symbol of $\mathrm{K}[w \square]$. Then ll is defined by § 28 and $l(l \circ l) = ll \circ ll = (l \circ l)l$ by § 30. Now suppose that the formulas of § 30 have been established for all symbols of $\mathrm{K}[l \circ]$ up to and including a certain a and for every b and c.

$$\begin{aligned} \text{Then } (l \circ a \circ b)c &= \{l \circ (a \circ b)\}c \text{ by associative law,} \\ &= lc \circ (a \circ b)c \quad \text{by hypothesis,} \\ &= lc \circ (ac \circ bc) \quad \text{by hypothesis,} \\ &= lc \circ ac \circ bc \quad \text{by associative law,} \\ &= (l \circ a)c \circ bc. \end{aligned}$$

Similarly in every other case when a, b, c are replaced by $l \circ a$, $l \circ b$, $l \circ c$ respectively. But by definition $l(l \circ \mathrm{N}a) = ll \circ a \square \mathrm{N}a$ and so on. Therefore by strict induction the formulas hold for every a, b, c belonging to the same $\mathrm{K}[l \circ]$. Hence by *Prop. V* the theorem is true in this case. If h, k, l denote different members of $\mathrm{K}[w \square]$ the definition gives

$$(h \circ k)l = hl \circ kl \text{ and } h(k \circ l) = hk \circ hl.$$

Then the proof is completed inductively by the same reasoning as above.

32. *Corollary 1.* The higher rule is a C-rule and by it unequals with equals give unequals. For since our symbols form a well ordered set according to the lower rule, we have $\mathrm{a} = \mathrm{b} \circ \mathrm{x}$ for any two unequal elements a, b and therefore $\mathrm{ac} = \mathrm{bc} \circ \mathrm{xc}$ $\therefore$ not $= \mathrm{bc}$.

33. *Corollary 2.* The higher rule is associative.

For this property has been established for the elements $a, b, \ldots$ as members of $K[w \square]$. Therefore by the distributive law it holds generally.

34. *Corollary 3.* The higher rule is commutative. For this also has been proved in the class $K[w \square]$.

35. *Corollary 4.* The set of symbols defined in § 22 forms an abelian semigroup with reference to the higher rule.

36. *Scholium.* Beginning with any $a > w$ we have abelian semigroups with regard to both rules, and so beginning with w for one or the other rule, but no part beginning with w forms a semigroup on both rules.

37. *Proposition XIV.* Necessary and sufficient conditions that a set M constitute an abelian semigroup with respect to each of two rules of combination, one distributive over the other, that M contain an arbitrary w and that M be as generated an ordered set, are the rules of combination thus far defined.

38. *Corollary.* The properties in question may be expressed by the following postulates:

A. There shall be a higher and a lower rule of combination.

B. Combinations of symbols shall be so ordered that those made by the lower rule precede combinations of the same symbols by the higher.

C. The higher rule shall be distributive over the lower.

D. There shall be a set M containing the symbol w.

E. M shall form an abelian semigroup for the lower rule.

F. M shall form an abelian semigroup for the higher rule.

G. Every successive set generated according to requirements shall be an ordered set.

39. *Scholium.* M is a well ordered set with no modulus.

40. We postulated two rules of combination for the symbol w, but only one for the lower symbol u. If we should postulate a higher rule for u by the same requirements as for w the resulting set of symbols would not differ as regards any group property from the w-set; the new abelian semigroups would be respectively holoedrically isomorphic with the former. If we postulate no higher rule the set $K[u\circ]$ still forms an abelian semigroup holoedrically isomorphic with $K[w\circ]$.

41. Instead let us set up postulates for the u class of symbols differing from those of § 38 only in adding another postulate H. The symbol u shall be a modulus for the higher rule, with the restriction on A–G that they shall not conflict with H. Then the class $K[u\square]$ reduces to the single element u, the abelian semigroup on the two rules reduces to the set $K[u\circ]$ and postulate B drops out, since every combination by the higher rule is found among those made according to the lower rule. This new set of postulates may be replaced by the equivalent set.

42. **a.** There shall be a higher rule defined from a lower by the inductive formulas $(u\circ a)b = ub\circ ab$ and $a(u\circ b) = au\circ ab$.

b. There shall be a lower rule defined inductively by the formula $u\circ a\circ b = u\circ(a\circ b)$.

c. There shall be a set M containing the symbol u.

d. The set M shall possess the group property for every combination $u\circ a$.

e. M shall be an ordered set.

f. There shall be a set M′ built up by postulates c, d and e on the symbol uu.

g. The sets M′ and M shall be identical.

43. *Proposition XV.* The set of symbols defined by the above seven postulates constitutes an abelian semigroup for each rule and has a modulus for the higher rule, which is distributive over the lower.

Proof. The abelian semigroup on the lower rule is established by *Prop. VII*, the distributive law follows from XIII and the other semigroup by § 35. It remains to prove the existence of a modulus.

Let $e = uu$. By postulate g, e must occur in the set $K[u\circ] : u, u', u'', \ldots$

Suppose $e = u''$. Then every member of $K[e\circ]$ will be found in $K[u\circ]$ but beyond u'. That is, u and u' are not in $K[e\circ]$, contrary to g. Therefore the only possibility of satisfying g is $e = u$. Since, then, $uu = u$, it follows by the distributive law that u is a modulus.

44. Postulates a . . . g define the natural numbers as ordinal symbols and by *XV* contain all the laws of their arithmetic. The cardinal numbers may be defined in a manner now familiar as names of classes, *e. g.*, "five" is the name given to the class of all well ordered sets which are ordinally similar to the set u, u', u'', u''', u^{iv}.

45. *Definition.* A group is a semigroup which contains, corresponding to every a, b, in it, symbols p and q such that $a \circ p = b = q \circ a$.

46. *Proposition XVI.* Every group contains a modulus with respect to its defining C-rule.

47. *Definition.* If a class which has a modulus u for its defining C-rule contains symbols a, $\bar{a}$ such that $a \circ \bar{a} = u = \bar{a} \circ a$ then a, $\bar{a}$ are said to be each the *inverse* of the other.

48. *Proposition XVII.* No member of a semigroup can have more than one inverse in the semigroup. For otherwise equals with unequals would give equals.

49. *Proposition XVIII.* Every group contains the inverse of every one of its members.

50. *Proposition XIX.* A semigroup with modulus is a group if it also contains the inverse of every one of its members.

Proof. $a \circ (\bar{a} \circ b) = a \circ \bar{a} \circ b = u \circ b = b$, and $(b \circ \bar{a}) \circ a = b \circ (\bar{a} \circ a) = b \circ u = b$, that is, § 45 is satisfied by $p = \bar{a} \circ b$ and $q = b \circ \bar{a}$.

51. *Proposition XX.* Given an abelian semigroup G with reference to a rule denoted by $\circ$, if in the class $[(m, n)] = C$ of pairs of elements of G we declare $(m, q) = (n, p)$ when and only when $m \circ p = n \circ q$ and set up a rule defined by the relation $(m, q) \odot (n, r) = (m \circ n, q \circ r)$, then 1) C will be a K-class, 2) $\odot$ a C-rule, 3) C an abelian group as regards the rule denoted by the sign $\odot$.

Proof. 1) Either $m \circ p = n \circ q$ or $m \circ p$ is not $= n \circ q$ by hypothesis and § 1. Hence either $(m, q) = (n, p)$ or (m, q) is not $= (n, p)$. Obviously the assertions $(n, p) = (m, q)$ and $(m, q) = (n, p)$ are identical in meaning, since this is so for $n \circ q$ and $m \circ p$. It remains to verify the euclidean postulate.

Suppose that $(m, q) = (l, r)$ and $(l, r) = (n, p)$.

Then $m \circ r = l \circ q$ and $l \circ p = n \circ r$.

Hence $(m \circ r) \circ (l \circ p) = (n \circ r) \circ (l \circ q)$.

But since G is an abelian semigroup $(m \circ r) \circ (l \circ p) = m \circ r \circ l \circ p = (m \circ p) \circ (l \circ r)$ and likewise $(n \circ r) \circ (l \circ q) = (n \circ q) \circ (l \circ r)$. Therefore $(m \circ p) \circ (l \circ r) = (n \circ q) \circ (l \circ r)$. Whence $m \circ p = n \circ q$ by definition of semigroup and finally $(m, q) = (n, p)$ by definition of equality.

2) Let $(m', q') = (m, q)$ and $(n', r') = (n, r)$. Therefore $m' \circ q = m \circ q'$ and $n' \circ r = n \circ r'$. Whence as under 1) $(m' \circ n') \circ (q \circ r) = (m \circ n) \circ (q' \circ r')$ and hence $(m' \circ n', q' \circ r' = (m \circ n, q \circ r)$. That is $(m', q') \odot$

$(n', r') = (m, q) \odot (n, r)$, or equals by equals give equals and $\odot$ denotes a C-rule.

3) The fundamental group property is secured by the definition. To prove the associative law we have $(m, n) \odot \{(p, q) \odot (r, s)\} = (m \circ p \circ r, n \circ q \circ s) = (m, n) \odot (p, q) \odot (r, s)$. Suppose that $(m', q') \odot (n, r) = (m, q) \odot (n, r)$. Then $(m' \circ n, q' \circ r') = (m \circ n, q \circ r)$ by definition. Hence $(m' \circ n) \circ (q \circ r) = (m \circ n) \circ (q' \circ r)$ and $(m' \circ q) \circ (n \circ r) = (m \circ q') \circ (n \circ r)$. Therefore $m' \circ q = m \circ q'$ by § 14. That is $(m', q') = (m, q)$ and similarly for the other form of this property. Thus C is a semigroup. But the element (m, m) is a modulus and to every (n, q) corresponds an inverse (q, n). Therefore by *XIX*, C is a group, which we will denote by G.

Finally $(n, r) \odot (m, q) = (n \circ m, r \circ q)$
$= (m \circ n, q \circ r) = (m, q) \odot (n, r)$.

Therefore by § 16, G is an abelian group.

52. The semigroup C and group G are connected by *Proposition XXI.* If we declare $(m \circ q, q) = m$ then $(m \circ q, q) \odot (n \circ r, r) = m \circ n$. For $(m \circ q, q) \odot (n \circ r, r)$ $= (\overline{m \circ q} \circ \overline{n \circ r}, q \circ r) = m \circ n \circ q \circ r, q \circ r) = m \circ n$.

53. *Scholium.* The group G contains a semigroup K holoedrically isomorphic with C in such manner that the rule $\odot$ becomes identical with the rule $\circ$ for K.

54. Obviously *Prop. XX* may be applied to the set of natural numbers in two different ways according as we use the semigroup on addition or that on multiplication. An essential distinction between these two procedures is furnished by

55. *Proposition XXII.* Given two rules of combination of which one is distributive over the other, no set of symbols can form a group with respect to both rules. For such a set would contain a modulus v for the lower rule (*XVI*). Then by the distributive law

$a = bx = b(x \circ v) = bx \circ bv = a \circ bv$ for every a. That is bv is also a modulus. But there can not be more than one modulus (*III*). Therefore $bv = v$ for every b so that when b' is not $= b$ we have $b'v = v = bv$ contrary to definition.

56. *Corollary.* It is not possible to define a set of symbols constituting a semigroup with modulus as regards both rules of § 55.

57. If *Prop. XX* is applied to the set of natural numbers and the rule of addition, we lose the semigroup on multiplication and can not recover it. This semigroup, in fact, is destroyed if we only annex a modulus for addition. Consequently we give the preference to the semigroup on multiplication and proceed with the group G obtained by applying *XX* to it.

58. *Proposition XXIII.* Given a higher rule distributive over a lower, and a set of symbols forming an abelian semigroup with respect to each rule, if we build the group G of § 52 with reference to the higher rule, then a necessary and sufficient condition of having a lower rule over which the higher shall be distributive and with regard to which G shall constitute an abelian semigroup is the formula of definition $(m, q) \circ (n, r) = (mr \circ nq, qr)$.

Proof. The necessity of this relation is obvious. To show that it is sufficient we first let $(m', q') = (m, q)$ and $(n', r') = (n, r)$ $\therefore (m'q = mq')$ and $n'r = nr'$. Therefore $m'r'qr = m'qrr' = mrq'r'$ and $n'q'qr = nqq'r'$. Hence $(m'r' \circ n'q)qr = (mr \circ nq)q'r'$ by distributive law and $\therefore$ by definition $(m'r' \circ n'q'q'r') = (mr \circ nq, qr)$ or equals with equals give equals and we have a C-rule. Similarly equals with unequals give unequals. For the associative law $(k, l) \circ \{(m, q) \circ (n, r)\} = (k, l) \circ (mr \circ nq, qr) = (kqr \circ lmr \circ lnq, lqr) = (kq \circ lm, lq) \circ (n, r) = (k, l) \circ (m, q) \circ (n, r)$.

Obviously $(n, r) \circ (m, q) = (m, q) \circ (n, r)$. Thus we have an abelian semigroup and the distributive principle follows by taking equals with equals.

59. *Definition.* The symbols of G shall be ordered according to the convention that (m, q) shall precede or follow (n, q) according as m precedes or follows n and by § 3.

60. *Scholium.* If a, b denote two natural numbers that have no common factor other than unity, the set of symbols (a, b) is simply ordered by the above definition. This set, which we will denote by R, may be taken as representative of G since every element of G is equal to some member of R.

61. *Definition.* The symbols constituting the set R are called absolute, rational numbers.

62. *Proposition XXIV.* Every two unequal elements of G are connected by a relation $h = g \circ x$ where h, g denote the given elements, g being that which precedes.

For $g = (m, q) = (mr, qr)$ and $h = (n, r) = (nq, qr)$ and mr precedes nq by hypothesis. Therefore $h = g \circ x$ is satisfied by $x = (d, qr)$ where $nq = mr \circ d$.

63. *Corollary.* The set R is simply ordered according to the lower rule denoted by $\circ$.

64. *Proposition XXV.* If we exclude the modulus u the remaining symbols of R form a set which is simply ordered according to the higher rule. For let x denote any element of R which precedes and y any one which follows u.

Then gx precedes and gy follows g. That is, the trios u, g, gx and u, g, gy are ordered whether g precedes or follows u.

65. *Corollary.* The modulus u separates the remaining symbols of R into two simply ordered classes, neither of which contains either a first or last element.

66. It is possible to divide R into two ordered classes so that every element of R belongs either to the one or to the other, but neither class has either a first or a last member. For example, the class R_1 of all elements whose squares precede 2 and the class R_2 of all whose squares follow 2 together exhaust the set R, while neither R_1 nor R_2 has either a first or a last element. If we take 4 instead of 2, the element 2 is not included either in R or in R_2; it separates them. We usually make this separation in practice by selecting a well ordered set, *e. g.* according to the decimal scale. We take first the highest integer in R_1, then the highest number of tenths, hundredths and so on. Similarly we pick out first the lowest integer in R_2, then the lowest set of tenths, hundredths, etc.

67. *Definition.* A well ordered set which has no last element is called a series.

68. *Definition.* The series of all symbols of the well ordered set $1, 2, \ldots w, w+1, \ldots 2w, \ldots 3w, \ldots w^2, \ldots$ which precede the element a taken in the above order, is said to be a series of type a.

69. *Definition.* An ordered set which has no first nor last element will be called an unbounded set.

70. Given an unbounded set S ordered according to a rule with respect to which S constitutes an abelian semigroup, suppose that a series $K[f_n]$ of type w has been selected from among the elements of S and let W_1 denote the set of all symbols of S which follow, V_2 all which precede every f_i, let V_1 denote the set of all symbols s which precede W_1 and W_2 the set of all that follow V_2.

Then one of the sets W_1, V_1 must be unbounded and both may be. So also of V_2 and W_2.

71. In every one of the four cases when

W_1 has a first element or V_1 has a last

V_2 has a last element or W_2 a first

this element is called the limit of the set to which it belongs.

72. In case W_1 and V_1 or V_2 and W_2 are both unbounded we will introduce a new symbol f_w which we will call the limit of $K[f_n]$.

73. *Lemma.* If W_i and V_i are both unbounded for the series $K[f_n]$ the same is true for the series $K[a \circ f_n]$ where a is any symbol in S. For if either the W or V set corresponding to $K[a \circ f_n]$ had a limit l we could write $l = a \circ x$ and x would be a limit for the same W or V belonging to $K[f_n]$ contrary to hypothesis.

74. *Theorem.* If the series $K[f_i]$, $K[g_j]$ both divide S into two unbounded sets the same is true of the set $K[f_i \circ g_i]$.

Proof. Let $h_1 = f_1 \circ g_1$, $h_2 = f_2 \circ g_2$, ...

$h_{w+1} = f_2 \circ g_1$, $h_{w+2} = f_2 \circ g_2$, . . .

$h_{2w+1} = f_3 \circ g_1$, $h_{2w+2} = f_3 \circ g_2$, . . .

Then by the Lemma each of the series $K[h_j]$, $K[h_{w+j}]$, $K[h_{2w+j}]$, . . . $K[h_{iw+j}]$, . . . divides S into two unbounded sets. Hence the same is true of the set h_1, h_{w+2}, h_{2w+3}, . . . h_{iw+i+1}.

75. On the basis of the preceding theorem and definitions is built up, by laying down the usual definitions of equality, order and two rules of combination, the set X of limits of series $K[f_n]$ selected out of R. The set X forms an abelian group with reference to the higher rule of combination and an abelian semigroup on the lower rule. The higher rule is distributive over the lower and possesses the modulus u. X may be simply ordered according to the lower rule and is divided by the modulus u into two unbounded sets each

simply ordered according to the higher rule. X is itself an unbounded set and contains a sub set holoedrically isomorphic with R as regards each rule of combination. Thus the whole set of absolute numbers is deduced from the set of absolute rational numbers by using only principles of order.

76. The preceding development does not take account of the possibility that $K[f_n]$ may not really divide S. There may be no symbol which follows every f_i. Then W_1 is an empty class and V_1 is taken to coincide with S. Thus if $K[f_n]$ and $K[g_n]$ are both series of this kind, they both have the same W and V, whence if new symbols f_w, g_w were to be introduced the principle of definition used in § 75 would lead us to declare $f_w = g_w$. Therefore we assign to the totality of the sets $K|f_n|$ for which W_1 is an empty class a single symbol Z which is to follow every element of X. Then following § 75, $Z \circ g_w$ is to be defined as h_w where $h_n = f_n \circ g_n$ and $f_w = Z$. But for $K[h_w]$, W_1 is empty, therefore $Z \circ x = Z \circ g_w = h_w = Z$. That is, $Z + x = Z = x + Z$, $Zx = Z = xZ$, and similarly $Z + Z = Z = ZZ = Z^3 = Z^N$. The associative, commutative and distributive laws still hold as well as the first group property, but the semigroup is destroyed for both rules by violating the law of equals with unequals.

77. It is also possible that no symbol in S precedes every f_i. Then V_2 is an empty class and W_2 coincides with S. On the same principle as in § 76 we assign to the totality of series of this kind a single symbol v to precede every x and define $v \circ x = h_w$ where $h_n = f_n \circ g_n$, $g_w = x$, $f_w = v$. And now the two rules must be distinguished. Since no x precedes every f_i the sets $K[f_n + g_n]$ and $K|g_n|$ separate S into the same V and W. Therefore $v + x = h_w = g_w = x + v$. And for the same reason no s can precede every $f_n\, g_n$

$$\therefore \quad vx = (fg)_w = v = xv$$

Also in like manner $\mathrm{v} + \mathrm{v} = \mathrm{v} = \mathrm{vv}$.

In the set $\mathrm{K}[\mathrm{h}, \mathrm{v}]$ the associative, commutative and distributive laws and the first group property are all preserved. The law of equals with unequals is violated for multiplication but still holds for addition, for which v is a modulus.

78. Since the introduction of infinity would destroy both semigroups while zero destroys only that on multiplication, it seems preferable to admit zero to our arithmetic and exclude infinity. But this removes the only objection to enlarging the addition semigroup into a group. This is effected at a stroke by applying *Prop. XX* to the set X and thus building an abelian group with reference to addition. In this group we then define a higher rule by the formulas

$$\bar{a} \odot b = \overline{ab} = a \odot \bar{b} \text{ and } \bar{a} \odot \bar{b} = ab.$$

This completes the system of real numbers, forming an abelian group on addition and, when we exclude its modulus, an abelian group on multiplication, which is distributive over addition. The system of real numbers can be simply ordered according to the lower rule, but the abelian group on the higher rule can not be similarly treated.

79. We might have proceeded by first applying XX to the natural numbers as a semigroup on addition. This yields the whole set of integers, positive, negative and zero. Omitting zero and defining by the law of signs we should have an abelian semigroup on multiplication. Applying XX to it, there results an abelian group composed of all rational numbers except zero. Then the introduction of the limits would supply the whole set of irrational numbers and close by reintroducing zero.

80. As long as we postulate two rules of combination, one distributive over the other, and demand semi-

groups with reference to both, we are led inevitably to the transfinite types in case we impose no restriction on the multiplication table and, if we require a modulus, to the natural numbers, whence the system of real numbers results from the attempt to build groups. The two rules of combination connected by the distributive property may be regarded as defining arithmetic up to date; it has not yet been found profitable to postulate in any other way. Then transfinite arithmetic is distinguished by postulating the semigroups and finite arithmetic by the postulate of the modulus.

81. The real numbers form the most general finite system with a single unit and must enter into every system with more than one principal unit, one unit must always be the modulus. Systems with two or more principal units have been studied exhaustively by Weierstrass. Here the abelian group on the lower rule is postulated universally.

With two principal units, if we exclude the modulus of the lower rule, the necessary and sufficient condition of an abelian semigroup on the higher rule is the existence in the system of a symbol i such that $ii = \bar{u}$. Thus the common complex numbers form the most general system with two principal units satisfying the postulates for real members.

82. Within the system of common algebra are distinguished different number bodies each built on a root of a given algebraic equation as a unit. The algebraic numbers of a given body form an abelian group on addition and, excluding zero, an abelian group on multiplication, and must contain a given symbol, the root of the given algebraic equation. For example the system of common complex numbers is the number body which contains a root of the quadratic $x^2 + 1 = 0$.

83. An integral algebraic number satisfies an equation

$$x^N + a_1 x^{N-1} + \ldots + a_i x^{N-i} + \ldots + a_n = 0,$$

where every a_i is an integer. A good illustration of the serviceableness of the group theory process for arithmetic is the theorem: The whole numbers of a quadratic number body form an abelian group on addition and, if we exclude zero, an abelian semigroup with regard to multiplication.

84. A more striking illustration is furnished by Dedekinds' "Ideals." An ideal is defined as a combination of the whole numbers of a number body possessing the first group property for both addition and multiplication. The ideals of a given body form an abelian semigroup with respect to multiplication. There is no addition of ideals.

85. Similar applications are found in the expression of an ideal by aid of its basis and in the cognate formulation of the transformations of a collineation in space of N dimensions. The latter question resolves itself into that of complex numbers with N principal units. Here it is no longer possible to preserve even the restricted abelian semigroup on the higher rule.

Not only is the abelian character lost, as in quaternions, but the semigroup property may be violated on account of the possibility of a combination by the higher rule being zero when none of the factors is zero.

86. It is now possible to describe more concisely the relation of the transfinite arithmetic to common arithmetic. The symbols defined in § 22 form a set of numbers with infinitely many principal units w, w^2, w^3, . . *i. e.*, the principal units form a series of type w. In finite arithmetic, it is true, the symbols u^2, u^3, . . . all belong to the set generated from u as principal unit, but if we allow this analogy in the transfinite system there will be still more new symbols.

For just as the removal of restrictions on the higher rule carried our series of symbols beyond the finite set of type w so the introduction of a still higher rule, corresponding to involution, will extend it beyond the set defined in § 22 if we impose no restrictions on the new rule except that of order and the special form of the group property.

87. Let us postulate a rule expressed by a^b higher than the two preceding, that is, the combination a b shall precede a^b. To distinguish we will now call $a\,b$ the lower and $a \circ b$ the lowest rule.

The lower rule has been shown to be associative and commutative for the set $K[w\,\square]$, which forms an abelian semigroup with regard to it. A set of symbols shall be generated from w by the higher rule in accordance with the postulate of order and the restricted form of the group property for w^a, where a is a previously defined element. By Prop. I this set forms a series $w, w^w = w', w^{w'} = w'', w^{w''} w''', \ldots$ and now w' must follow every w^N $(N = 2, 3, \ldots)$ [w^N is not a combination by rule, the N is a mere index]. Then by § 22 we obtain a series holoedrically isomorphic with the set there considered if to the lowest and lower rules respectively we make the lower and higher correspond.

88. The articulation of the two lower rules in §§ 28–35 leaves room for much freedom in definition. There we defined the lowest rule first and made the lower depend upon it. Here we have already defined the lower rule to the extent implied in the application of § 22. This, however, involves no further restriction than the inductive associative formula of § 19. But the definition from the lowest rule carries this property with it.

It is therefore permissible to define the lowest rule so that the lower shall be distributive over it. Then by the process of § 22 new combinations are made by the higher and lowest rule together, from each series

of type w in the present set of symbols a new set of the same type as that in § 22. And all new series being fitted in by the nature of the process between consecutive elements of the series previously defined we have at every stage a totality of symbols forming a series.

89. The group properties of the two lowest rules are preserved throughout the preceding process. On every new symbol not obtained from the preceding by postulating them is built up a set forming abelian semigroups on both the lower and lowest rules, the new symbol playing the part of a unit precisely like w in the series of § 22, and in this set the lower rule is distributive over the lowest. This process is to be continued as long as new symbols can be obtained by it.

90. There remains one more possibility of building new symbols by introducing a combination defined one way of the higher symbols with those of the series $K[u\circ]$ according to the lowest rule. Thus to every a in the higher series will be assigned a w-series a, $a\circ u$, $a\circ u'$, $a\circ u''$, $a\circ u'''$, . . .

This series fits in between a and its next following element in the former set so that we still have a totality forming a series.

91. For completeness it is convenient also to define combinations one way of certain symbols with those of $K[u\circ]$ according to the other two rules, viz.: those expressions in which the symbols of $K[u\circ]$ have already been used as marks or indices, *e. g.*:

$u'w$, $u''w$, $u'''w$, . . . $w^{u'}$, $w^{u''}$, $w^{u'''}$, . . .
$u'w'$, $u''w'$, $u'''w'$, . . . $w'^{u'}$ $w'^{u''}$, $w'^{u'''}$, . . .

Expressions $u\circ a$, $u'\circ a$, . . . u'^{w}, u''^{w}, . . . $u'^{w'}$, $u''^{w'}$, . . . are not defined, neither are there any combinations according to the higher rule except the set w^{a} where a belongs to this set. The higher

rule has not been made a rule of combination for any whole set. This would involve assigning properties to it and completing the arithmetic of the symbols we have defined, which is beyond the present purpose. It is enough to have them in series; that forms the foundation of their arithmetic.

92. The mere existence of the series of symbols developed in the preceding §§ is enough to solve a great variety of problems arising in analysis, the nature of which will be illustrated by comparing this series with series consisting of absolute numbers. The natural numbers form a series of type w if they are arranged in the order of their genesis. It is, however, possible to arrange them in series of higher types for *e. g.* the odd numbers alone form a series of type w. If to this we add on the even numbers successively we obtain series of types

$$w+1,\ w+2,\ \ldots\ w+\mathrm{N},\ \ldots$$

and thus the whole set is ordered in type $2w$.

If we order the set R of § 60 as follows:

$$\begin{array}{l} (1,1),\ (2,1),\ (3,1).\ (4,1),\ \ldots \\ (1,2),\ (3,2),\ (5,2),\ (7,2),\ \ldots \\ (1,3),\ (2,3),\ (5,3),\ (7,3),\ \ldots \end{array}$$

we have a series which can be put ordinally into one to one correspondence with the series $1, 2, \ldots$ $w+1,\ w+2,\ \ldots\ 2w,\ 2w+1,\ \ldots$ *i. e.*, is of type w^2.

The same set of numbers (the common fractions) can, however, be arranged in a series of higher type. For every finite, simple, continued fraction is equal to some common fraction and conversely. Now let us order the finite continued fractions according to the values of their successive quotients

$$q_1,\ q_2,\ q_3,\ \ldots\ q_{\mathrm{N}}.$$

Thus the continued fractions containing each a single quotient form a series of type w. Then from each

member of this series by adding a second quotient results again type w. Therefore the fractions of the form $\frac{1}{q_1+}\frac{1}{q_2}$ may be arranged in series ordinally similar to 1, 2, . . . $w+1$, . . . $2w$, . . . $3w$, . . . *i. e.* of type w^2. Annexing a third quotient replaces each element of this set by a series of type w so that we include types w^2+1, w^2+2, . . . $2w^2$, . . . $3w^2$, . . . *i. e.* the class of fractions $\frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3}$ is ordered in type w^3.

Therefore the whole set of simple continued fractions, comprehending types w, . . . w^2, . . . w^3, . . . w^N, . . . constitutes, as ordered, a series of type $w^w = w'$.

93. Now it is easy to arrange the natural numbers also in a series of type w' as follows. We know that the class of prime numbers can be put into one to one correspondence with the whole set of natural numbers (the number of primes is infinite).

Therefore we can set up a one to one correspondence ordinally between the prime numbers and the simple continued fractions with a single quotient. By the same reasoning the continued fractions with two quotients are shown to be ordinally similar to the class of product of two primes, *i. e.*, if to every quotient q_1 we assign that prime p_1 for which q_1 is the ordinal number in sequence (so that *e. g.* to the quotient 7 we assign 17) and likewise for a second prime p_2 and quotient q_2, and so on.

Proceeding in this way the class of all products of N primes is exhibited as ordinally similar to the class of continued fractions $\frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_N}$. But the class of all products of primes is the whole set of natural numbers. Therefore by § 92 the natural numbers may be arranged in series w'. Q. E. D.

94. In other words we have here a method whereby

the elements of any w-series may be rearranged so as to produce a series of type w^w. Applying this method successively to the w-series in the preceding we obtain in place of w, $2w$, . . . w', $2w'$ . . . that is, instead of w^2 we get a type ww'. Then w^2+w, w^2+2w, . . . are replaced by $ww'+w'$, $ww'+2w'$, . . . making in all type $2ww'$ replacing the former $2w^2$.

Then come in succession types $3ww'$, $4ww'$, . . . so that the original w^3 expands into a type w^2w'. It is clear that in this way we retrace precisely the steps of the process of § 22, building up on each new symbol the abelian semigroups according to the lower and lowest rules.

Therefore by successive applications of the method it will be possible to rearrange the natural numbers in series of types as high as any of the set hitherto defined. This result may also be stated. The transfinite ordinal series so far defined may each be put into one to one correspondence with the set of natural numbers.

95. The symbols w, $w+1$, $w+2$, . . . *i. e.*, the transfinite symbols, are said to form the second ordinal class, the finite symbols constituting the first class. The latter class was said to be of type w, where w is the symbol following next after all the finite symbols of $K[u\circ]$. Likewise we shall say that the first and second ordinal classes together form a series of type W, introducing this new symbol without assigning to it any properties except that it shall follow next after all symbols of the second ordinal class, *i. e.*, after all the set $K[w^a] : w, w', w'', \ldots$

96. We see that the ordinal symbols play a double role. They were defined as symbols forming a well ordered set. But to each symbol which has no immediate predecessor corresponds a series of which it is the type, viz.: the series of all its predecessors or any ordinally similar series. The arrangements of the nat-

ural numbers in § 94 are only a part of these series. But they are also only a part of the permutations of the whole set of natural numbers. Other permutations are obtained by the following *Lemma.* From any w series of symbols may be obtained a set of permutations of the symbols forming a series of type $w' = w^w$.

Proof. Let a_1, a_2, . . . a_N, . . . denote the given symbols in the given order. Without changing the order of the higher a's put a_1 successively in every subsequent place. This set of permutations is obviously of type w.

From every one of them by repeating the process on a_2 we obtain again a w-series ∴ in all a series of permutations of type w^2. Repeating the process with a_3, a_4, ... successively we have a series of permutations whose type is w'. Q. E. D.

97. Thus the arrangements of the natural numbers furnish a series of type not lower than W. For by § 96 we first deduce from the series 1, 2, 3, ... *i. e.*, the normal order, a set of permutations forming a series of type w'. That is, to every ordinal symbol preceding w' is assigned a permutation, and *vice versa.* Then by § 94 arrange the natural numbers in series w' which obviously is a different permutation from any of the preceding ∵ it can not be obtained by the process of § 96. Now repeating the method of § 96 on each w-series of this permutation we use up all ordinal symbols between w' and that which results from it by a second application of § 94. This holds step by step as long as the latter process can be carried on. Thus to every ordinal symbol in succession preceding W is assigned a new permutation. That is, we have a set of permutations of all the natural numbers forming a series whose ordinal type is W.

98. The process just described for making permutations of all the natural numbers can not yield a series

of type higher than W since, as we have seen, it generates precisely the series W of ordinal symbols. That there are types higher than W is obvious, for we can proceed with W just as with w to generate new e-sets and new abelian semigroups, and there is no limit to the possibilities in the way of still higher symbols and rules. Now it is quite conceivable that there may be further permutations of natural numbers not obtainable by the above process. As a first step toward investigating this question let us consider the simpler one whether the natural numbers themselves can be arranged in a series of ordinal type higher than W. For this purpose we will establish the following

Lemma. Permutations of the natural numbers can be made by the process described and ordered so as to form a series of type higher than any series of all the natural numbers, however arranged. For by the process in question we can always form a new permutation which differs from the first permutation in at least its first element, from the second in at least its second element . . . . . from the $(w+1)$st in at least its $(w+1)$st element, and so on, therefore is not included in any set of permutations ordinally similar to any possible arrangement of all the natural numbers. From this Lemma we readily obtain the

Theorem. Every possible arrangement of the natural numbers is a series of the second class. For by the Lemma to every such arrangement in series can be assigned a set of permutations forming a series of higher ordinal type. But by § 97 the process by which this is effected yields a set of permutations forming a series of type W. Therefore every possible arrangement of natural numbers in series is of type lower than W, therefore belongs to the second ordinal class (being *ipso facto* $\geqq w$). Q. E. D.

99. The arrangement of the natural numbers in series of type higher than w finds application in the study of infinite continued fractions. By aid of the euclidean algorithm for greatest common divisor every absolute irrational number less than unity can be expressed as an infinite continued fraction

$$\frac{1}{q_1+}\,\frac{1}{q_2+}\cdots\cdots\frac{1}{q_N+}\cdots\cdots$$

and conversely. The class of infinite simple continued fractions may therefore be taken as the representative of the class of irrational numbers between zero and 1.

An infinite continued fraction can not be obtained from a finite one merely by annexing quotients; it can only be described by assigning a law which determines q_N for every value of N. An infinite continued fraction may be formed, *e. g.*, by the law that every quotient shall be 2. It is sufficient, however, to consider the class in which the quotients are all different; this can be put into one to one correspondence with the whole class. Accordingly we are concerned with the class of all possible permutations of all the natural numbers. These permutations may be examined in the same way as the permutations of a finite set by imagining a framework of places to be filled, but the number of places is infinite. Moreover, we must provide for the possibility of filling the places in a series of order higher than w. Thus an infinite continued fraction can be formed by filling the even places successively with the odd numbers in their natural order and the odd numbered places with even numbers in the same way, *i. e.*,

$$\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{6+}\,\frac{1}{5+}\cdots\cdots$$

or by filling the odd numbered places with primes and the even places with composite numbers. Or we can select first the places whose indices are primes, then the indices which are products of two primes, three

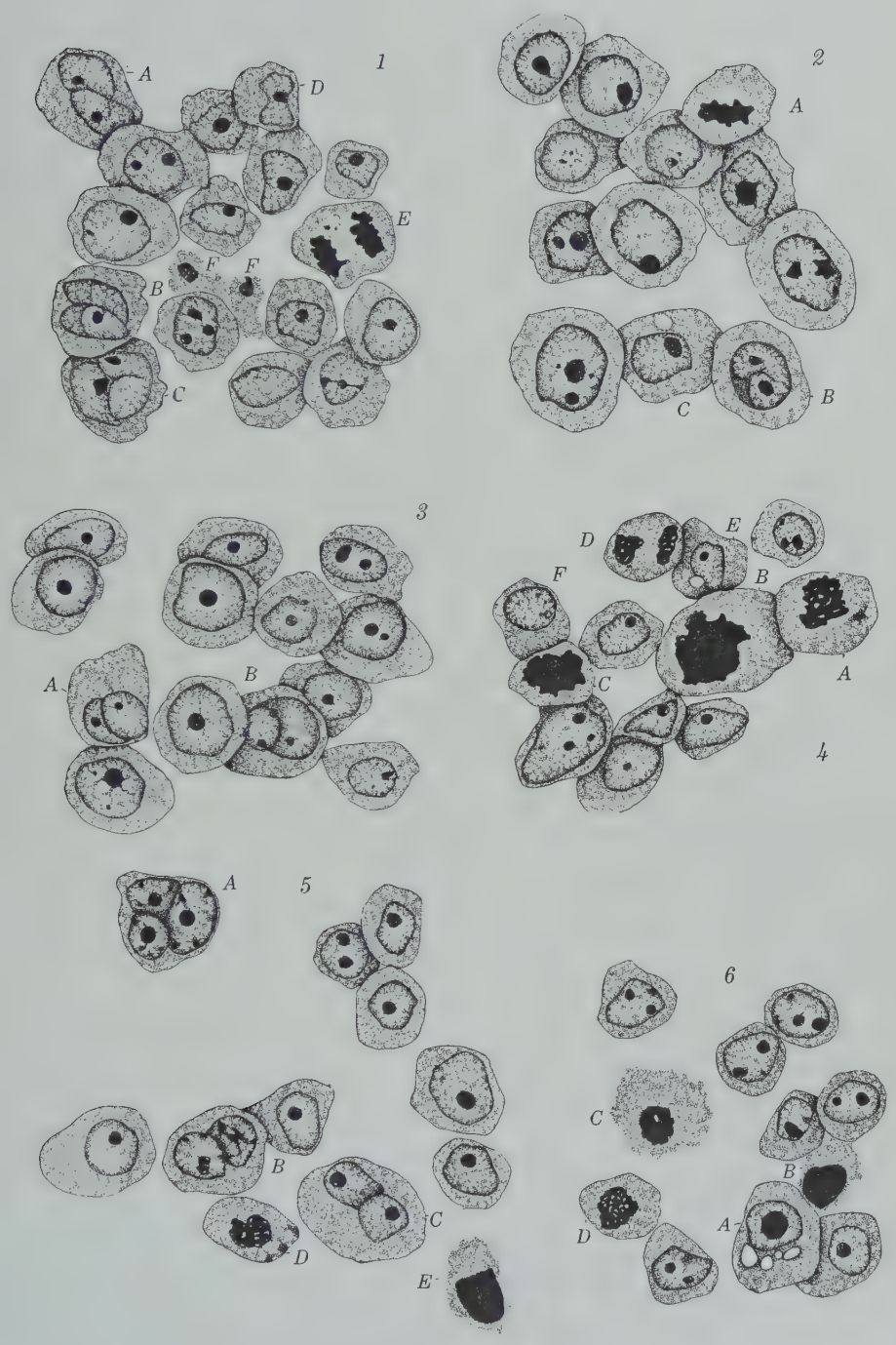
primes, and so on, and fill these sets with the corresponding sets of numbers permuted in any way we please.

100. We have seen that every possible arrangement of all the natural numbers in a series is of type lower than W. Therefore the quotient places of our framework *in the order in which we propose to fill them* (or rather state how they shall be filled) constitute an ordinal type of the second class. That is, the *numbered places* in a permutation of all the natural numbers, arranged, however, *in the order in which they are to be filled*, form a series obtainable by the process of § 97.* Hence there can be no permutation of all the natural numbers not obtainable by this process, since such a permutation would be *eo ipso* different from all of those hitherto obtained and well ordered in some definite numbered places. In other words, this would mean that the series of the places as they are filled, or the series of numbers with which each place is to be filled, was not obtainable by the process of § 97. Therefore all possible permutations of the natural numbers form a series closely related to the series of type W.

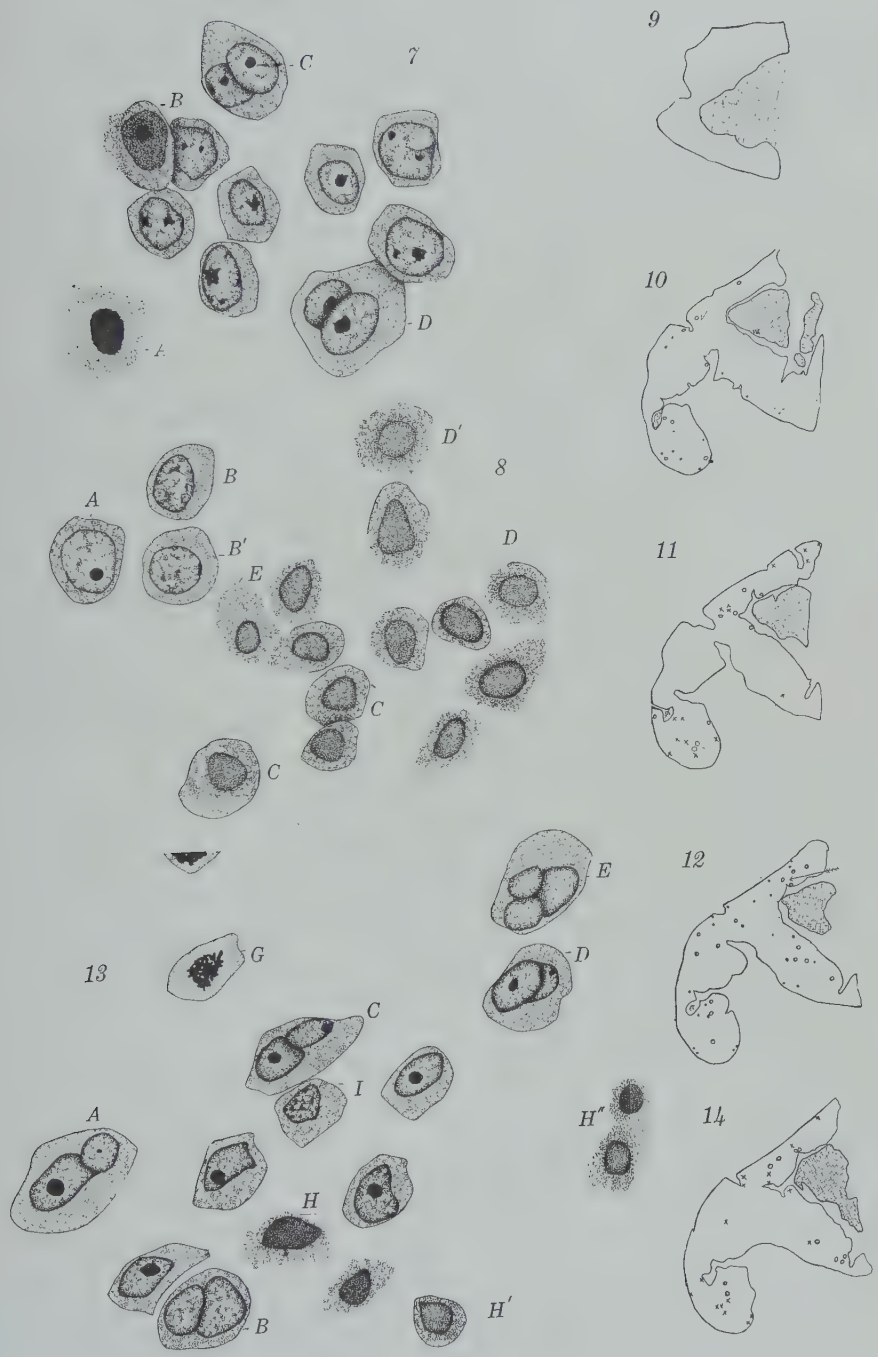
BIOGRAPHY.

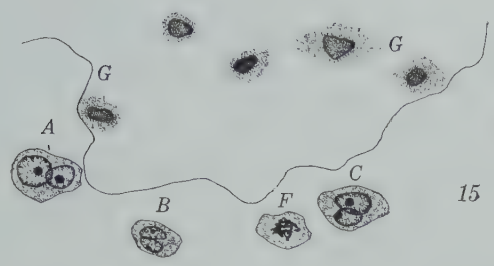
I, ARTHUR BOWES FRIZELL, member of the Protestant Episcopal Church, was born in Boston, July 14, 1865. My parents were Joseph Palmer Fessenden Frizell, civil and hydraulic engineer, and Julia Anna (Bowes) Frizell. My early education was chiefly at home and in a private school in Dorchester, Mass. I graduated from the high school in St. Paul, Minn., and spent three years at the Massachusetts Institute of Technology, where I afterwards served as assistant instructor in mathematics, 1888–'91. I received the degree of Bachelor of Arts from Harvard College in 1893 and that of Master of Arts from Harvard University in 1900. I served as instructor in mathematics at New York University 1895–'96, and at Harvard 1897–1906, when I resigned to study abroad. After three semesters at Göttingen, I returned to America, and was appointed, 1908, Professor of Mathematics in Midland College, Atchison, which position I resigned, 1909, to accept an instructorship in the University of Kansas.

*And the preceding statements apply *verbatim* to every series of values admissible in a given numbered place.



The University of Chicago Libraries.


7
C
B
D
A
8
D′
B
A
B′
E
D
C
C
9
10
11
12
E
D
13
G
C
I
A
H
B
H′
H″
14



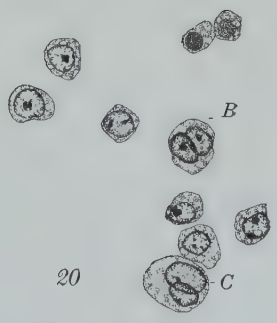
15

16

17

18

19



20

PLATE XLIX.

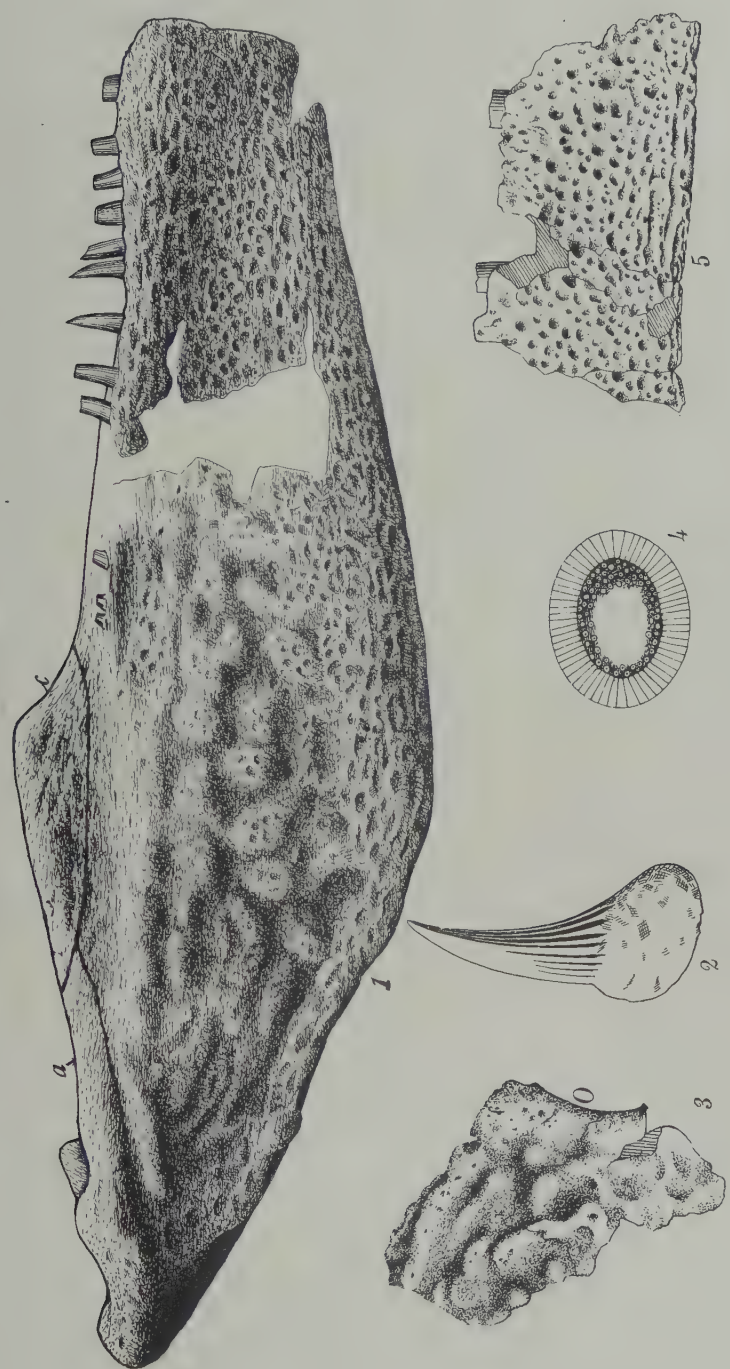
FIG. 1. External view of the right mandible of *Eryops willistoni*. × ½. Drawing by S. Prentice. *a* = articular; *c* = coronoid.

FIG. 2. Maxillary tooth of *Eryops willistoni*. × ½.

FIG. 3. Fragment of cranium, including portion of right orbit and showing very distinctly the suture bounding the anterior border of the right frontal. A portion of the supraorbital lateral line canal may be also detected. × ½. O = orbit.

FIG. 4. Cross section of mandibular tooth, showing pulp cavity and dental tubules. × 2.

FIG. 5. External view of anterior portion of the right mandible. × ½.



The University of Chicago Libraries.

PLATE L.

External view of the right coraco-scapula of *Eryops willistoni*. × ½. C = coracoid portion of the coraco-scapula; f = foramen for artery; L = cleithrum; O = humeral cotylus.

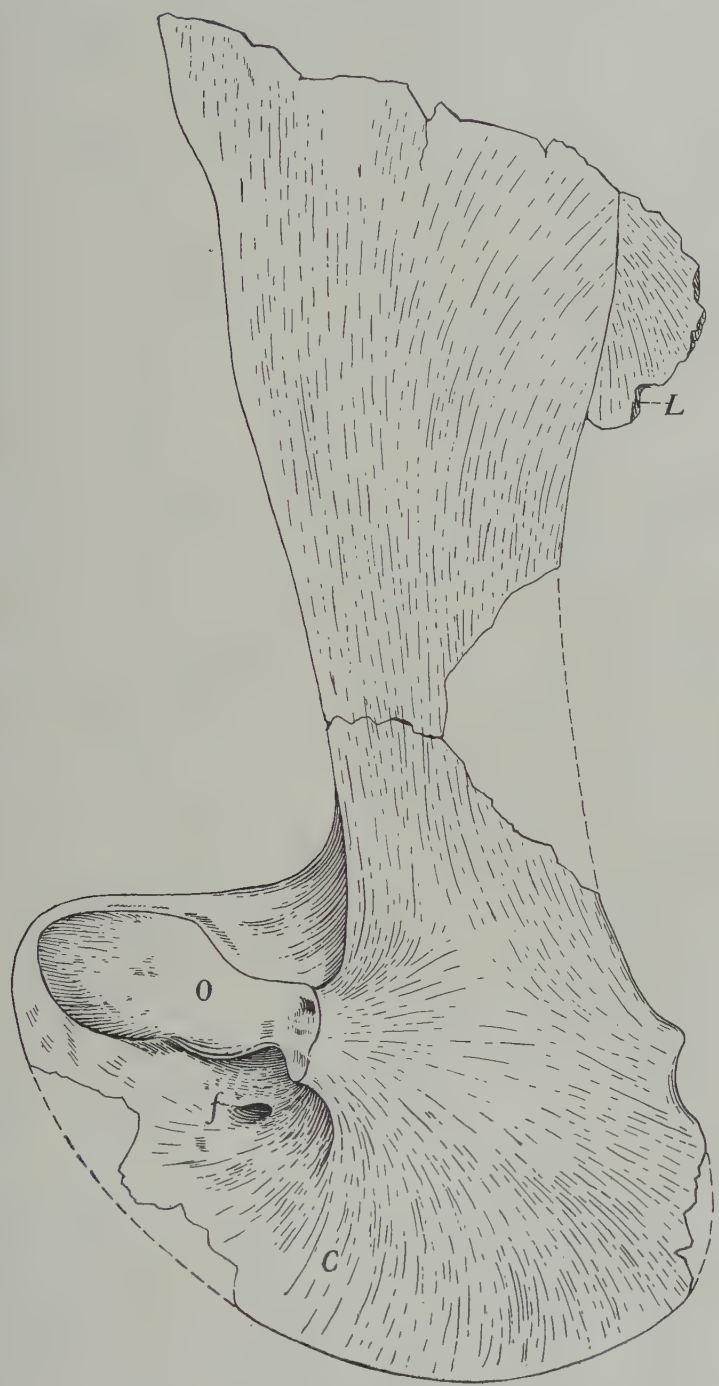

L
O
f
C

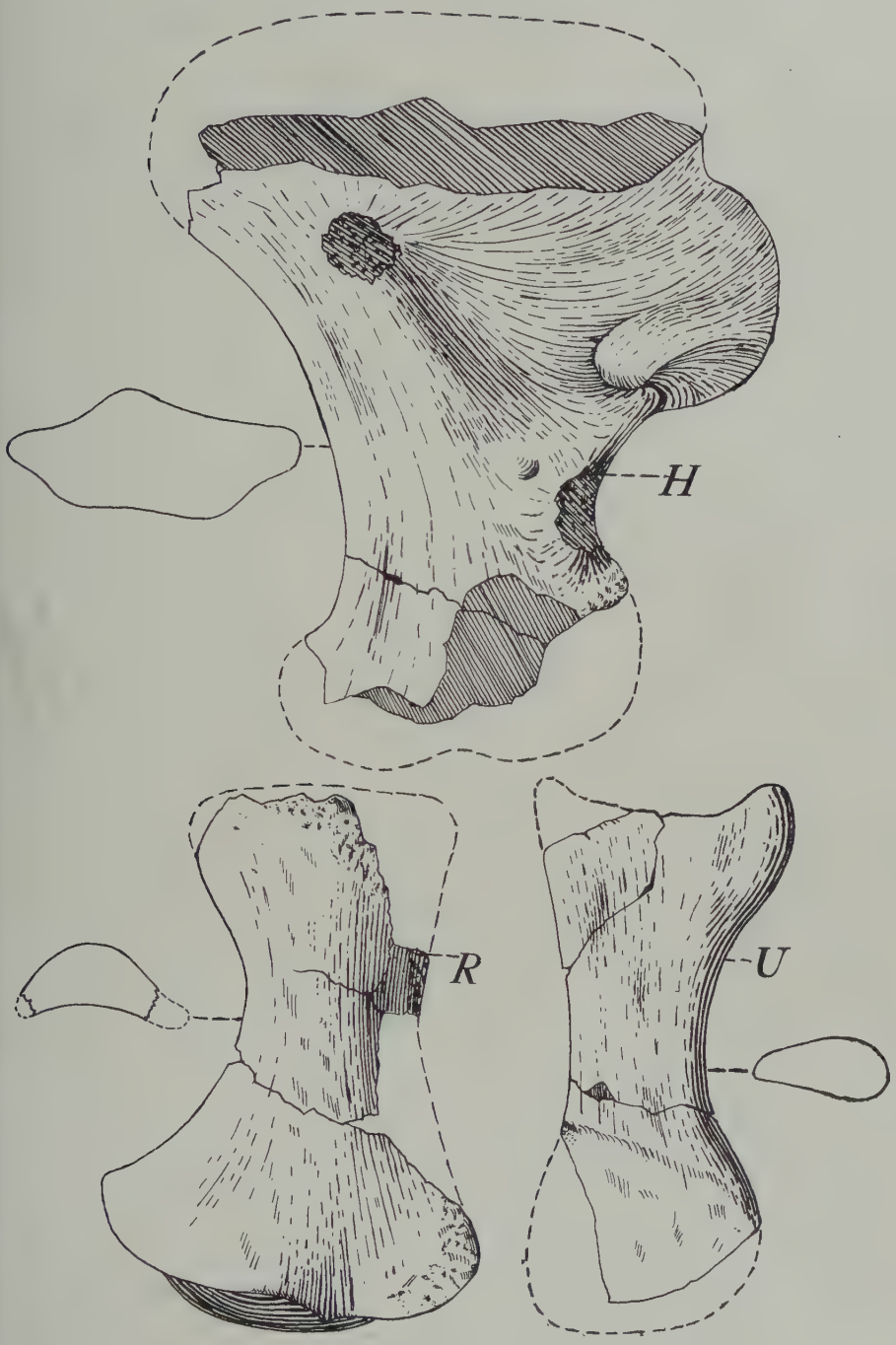
PLATE LI.

Internal view of coraco-scapula. C = coracoid portion; f = foramen; L = cleithrum. $\times$ ½.

The University of Chicago Libraries.

PLATE LII.

Arm bones of *Eryops willistoni*. × ⅔. *H* = humerus; *R* = radius; *U* = ulna. Cross sections of the various elements are shown at the sides.



The University of Chicago Libraries.

PLATE LIII.

FIG. 1. Posterior view of right clavicle of *Eryops willistoni*. × ⅔.

FIG. 2. Ventral view of right clavicle. × ⅔.

FIG. 3. Ventral view of the posterior projection of the interclavicle of *Eryops willistoni*. × ⅔.

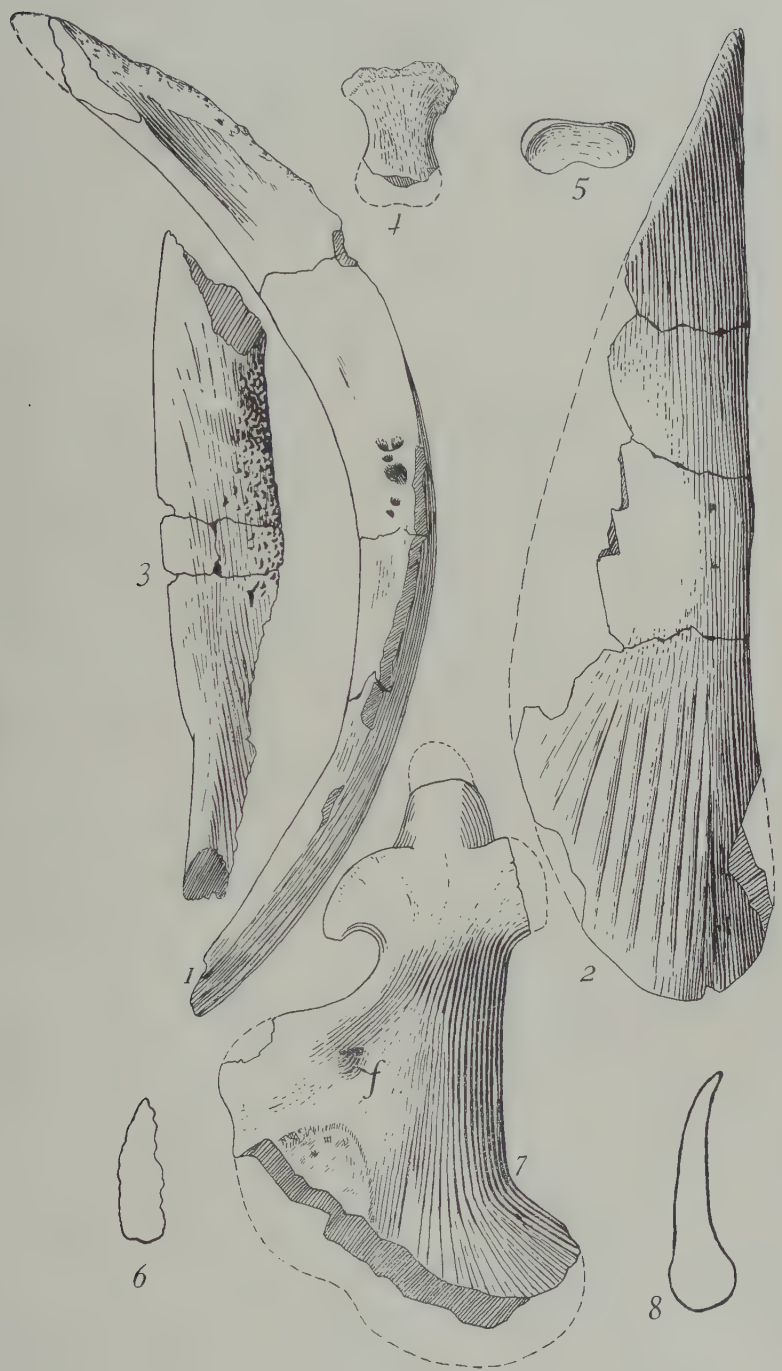
FIG. 4. Dorsal view of phalange (metacarpal ?). × ⅔.

FIG. 5. End view of same. × ⅔.

FIG. 6. Cross section of posterior projection of interclavicle.

FIG. 7. Anterior view of the right humerus. × ⅔. f = supracondylar arterial foramen.

FIG. 8. Cross section of the clavicle.



The University of Chicago Libraries.

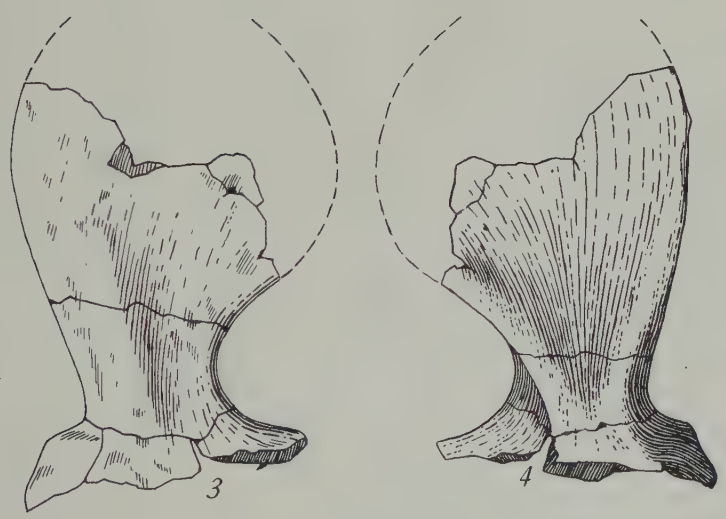
PLATE LIV.

FIG. 1. Posterior view of a dorsal vertebral spine, showing apical enlargement. × ⅔.

FIG. 2. Lateral view of three consecutive dorsal vertebræ. × ⅔.

FIG. 3. Ventral view of left sacral rib. × ⅔.

FIG. 4. Dorsal view of the same.



The University of Chicago Libraries.

PLATE LV.

The dorsal surface of the left ilium of *Stegopelta*, showing the dermal plates. × ¼.

The University of Chicago Libraries.

PLATE LVI.

FIG. 1. A large dermal spine which was possibly associated with the heavy bony girdles near the base of the tail. × ¼.

FIG. 2. One of the massive dermal plates of *Stegopelta,* which may have formed part of another girdle like the elements in figs. 6 and 7. × ½.

FIG. 3. The median dorsal scute between the ilia. × ¼.

FIGS. 4 and 5. The teeth of *Stegopelta.* × 1½.

FIGS. 6 and 7. Dorsal and posterior views of large bony girdle, which probably encircled the tail near the base. × ¼.

FIG. 8. Detail of the surface of the left ilium; one of the dermal scutes, one-half natural size.

FIG. 9. A heavy dermal plate, which may possibly have been associated with the element shown in figure 2 to form another girdle. (See Wieland, 1909, Amer. Jour. Sci., March, p. 251, fig. 6.) × ¼.

2
9
3
6
1
7
8
4
5

PLATE LVII.

Dorsal view of thirteen small dermal plates of *Stegopelta*, nearly natural size, showing the various shapes, sizes, and manner of ornamentation.

The University of Chicago Libraries.

PLATE LVIII.

FIG. 1. The external surface of the right pubis. × ¼.

FIG. 2. The right pubis seen from the edge. × ¼.

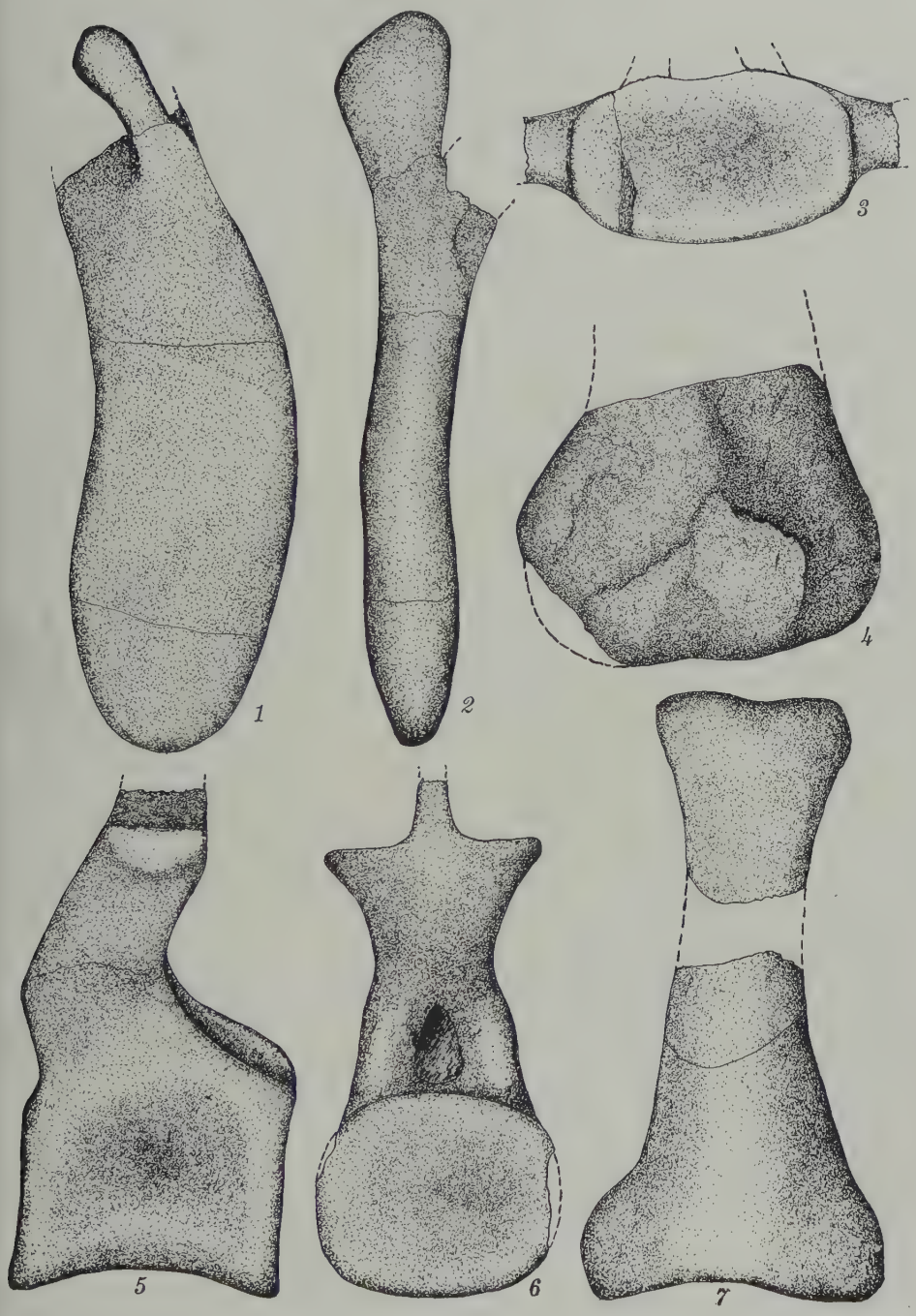
FIG. 3. The posterior view of an anterior caudal or a posterior sacral vertebra. × ¼.

FIG. 4. The lower portion of the right tibia, showing the firm union of the astragalus. One-half natural size.

FIG. 5. Lateral view of one of the dorsal vertebræ. One-half natural size.

FIG. 6. Posterior surface of a dorsal vertebra. × ½.

FIG. 7. A metatarsal of *Stegopelta* seen from above. × ½.



The University of Chicago Libraries.

PLATE LIX.

FIG. 1. Dorsal view of an anterior caudal vertebra. × ¼.

FIG. 2. A cross section of the sacrum to show the sacral enlargement. × ½.

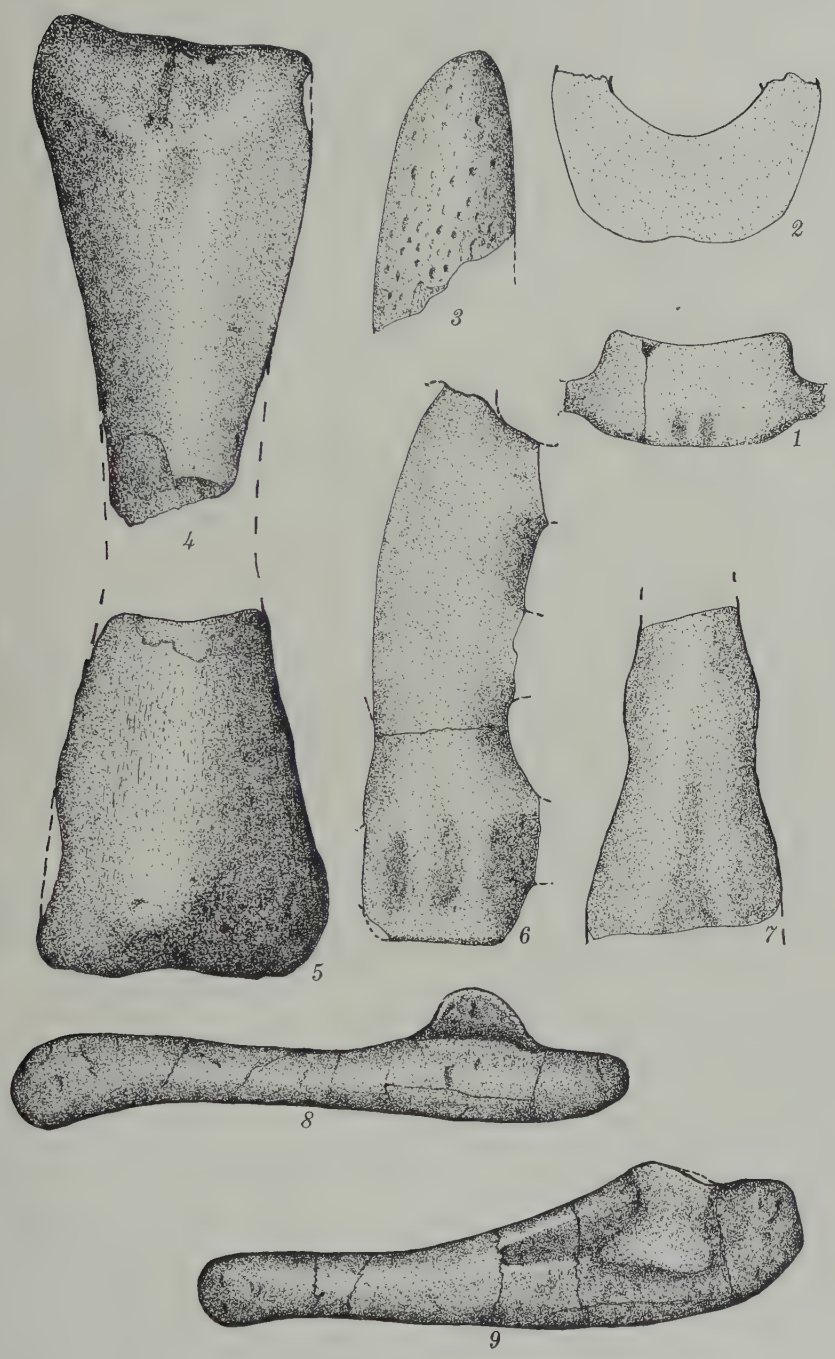
FIG. 3. A small dermal spine. × ½.

FIGS. 4 and 5. Upper and lower portions of right fibula. × ¼.

FIGS. 6 and 7. Dorsal surface of posterior and anterior segments of the sacrum.

FIG. 8. The left ulna seen from behind. One-tenth natural size.

FIG. 9. The left ulna seen from the side. One-tenth natural size.



The University of Chicago Libraries.

Plate LX.

The stomach of *Empo nepaholica* Cope from the Niobrara Cretaceous. The longitudinal muscular plicæ are indicated by heavy lines. × ½.

The University of Chicago Libraries.

PLATE LXI.

The specimen of *Thrissopater intestinalis* Moodie. × ⅔.

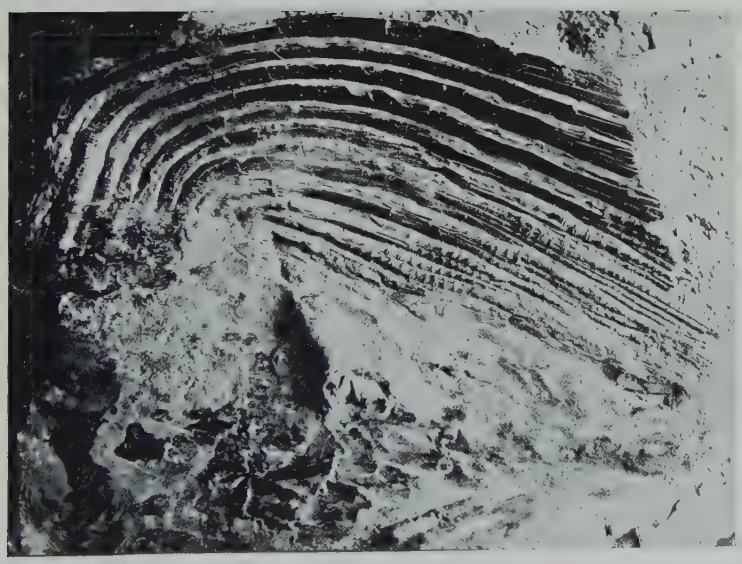
The University of Chicago Libraries.

Plate LXII.

Fig. 1. Photograph of the alimentary canal of the Buffalo fish, *Ictiobus bubalus* Raf., from the Kaw river, showing the similarity in the form of the intestines to that preserved in the fossil *Thrissopater intestinalis* Moodie. × ½.

Fig. 2. Photograph of the right pectoral fin of *Empo nepaholica* Cope. × ¾.

1



2

The University of Chicago Libraries.

PLATE LXIII.

Upper figure: Lateral view of female of *Prorocorypha snowi*, new gen. and sp. × 2.

Second figure: Lateral view of male of *Prorocorypha snowi*. × 2.

Left of three middle figures: Lateral view of apex of male abdomen of *Prorocorypha snowi*. × 4. (The supra-anal plate of the specimen from which the artist made the drawing is broken. It is long, as in the female.)

Middle of three middle figures: Dorsal view of head and pronotum of female of *Prorocorypha snowi*. × 2.

Right of three middle figures: Lateral view of apex of female abdomen of *Prorocorypha snowi*. × 4.

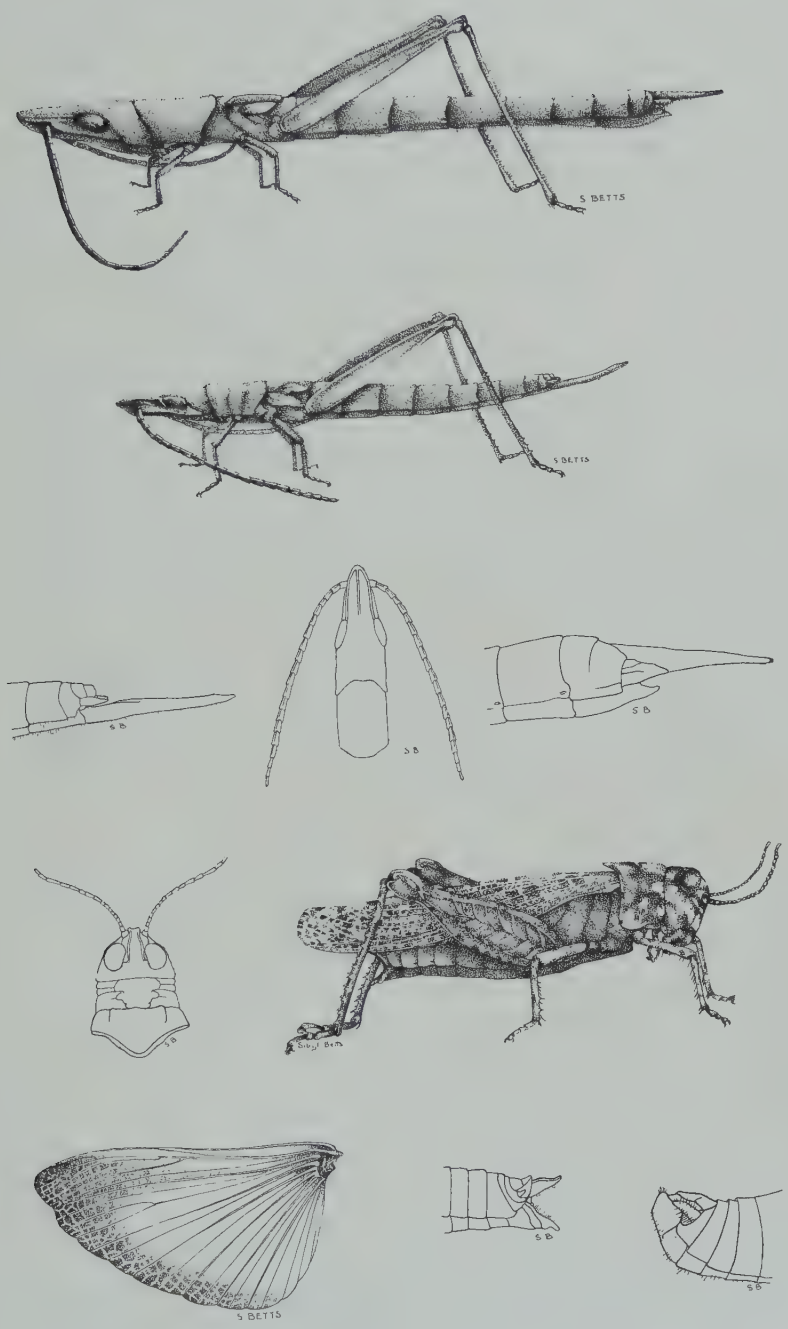
Next to bottom row, left figure: Dorsal view of head and pronotum of *Scirtetica ritensis*, new species. × 2.

Next to bottom row, right figure: Lateral view of type specimen of *Scirtetica ritensis*. × 2.

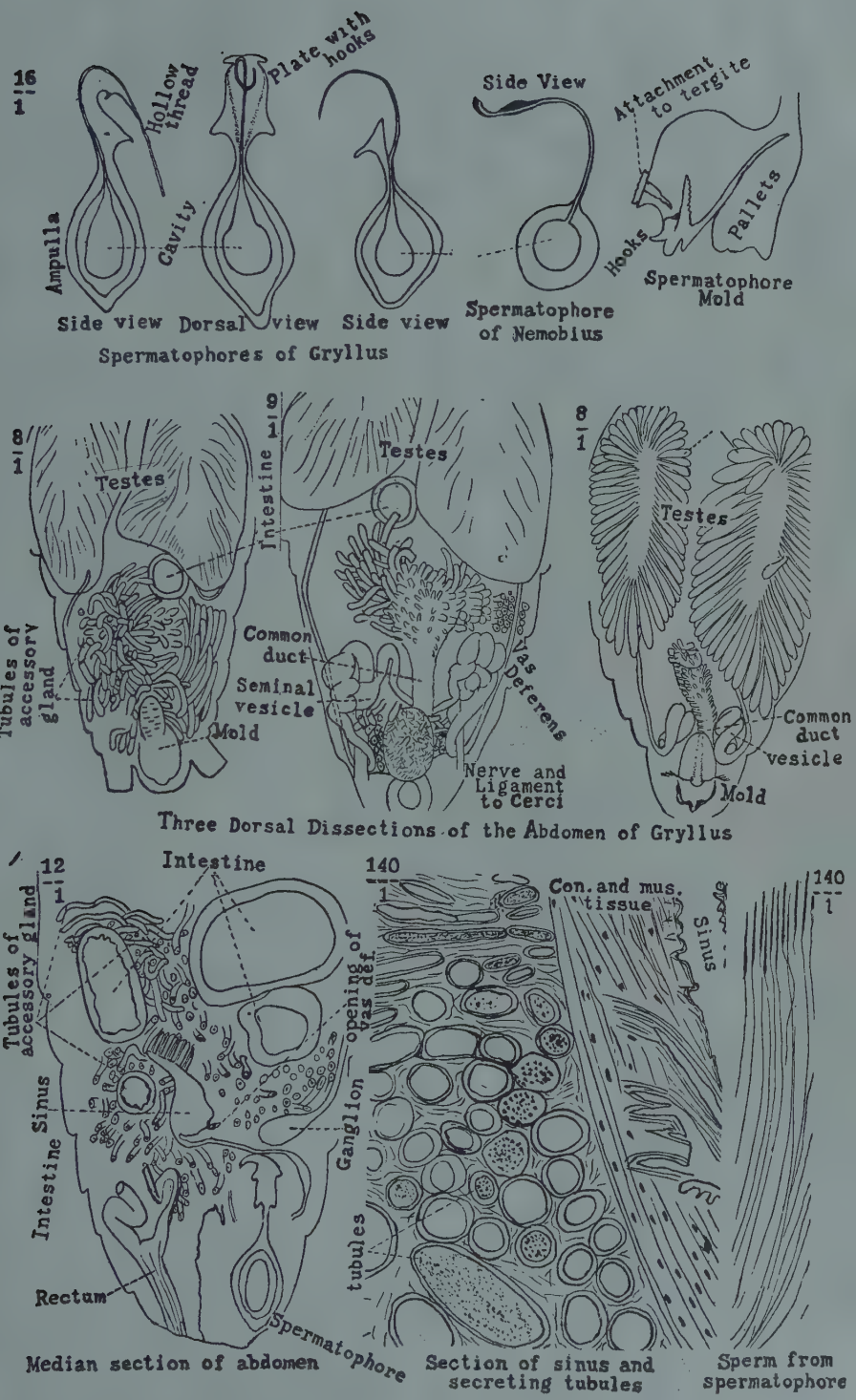
Bottom row, left figure: Wing of *Scirtetica ritensis*. × 2.

Bottom row, middle figure: Lateral view of apex of female abdomen of *Scirtetica ritensis*. × 2.

Bottom row, right figure: Lateral view of male abdomen of *Scirtetica ritensis*. × 2. (This figure was drawn by the artist from a specimen not studied by the author.)



The University of Chicago Libraries.



The University of Chicago Libraries

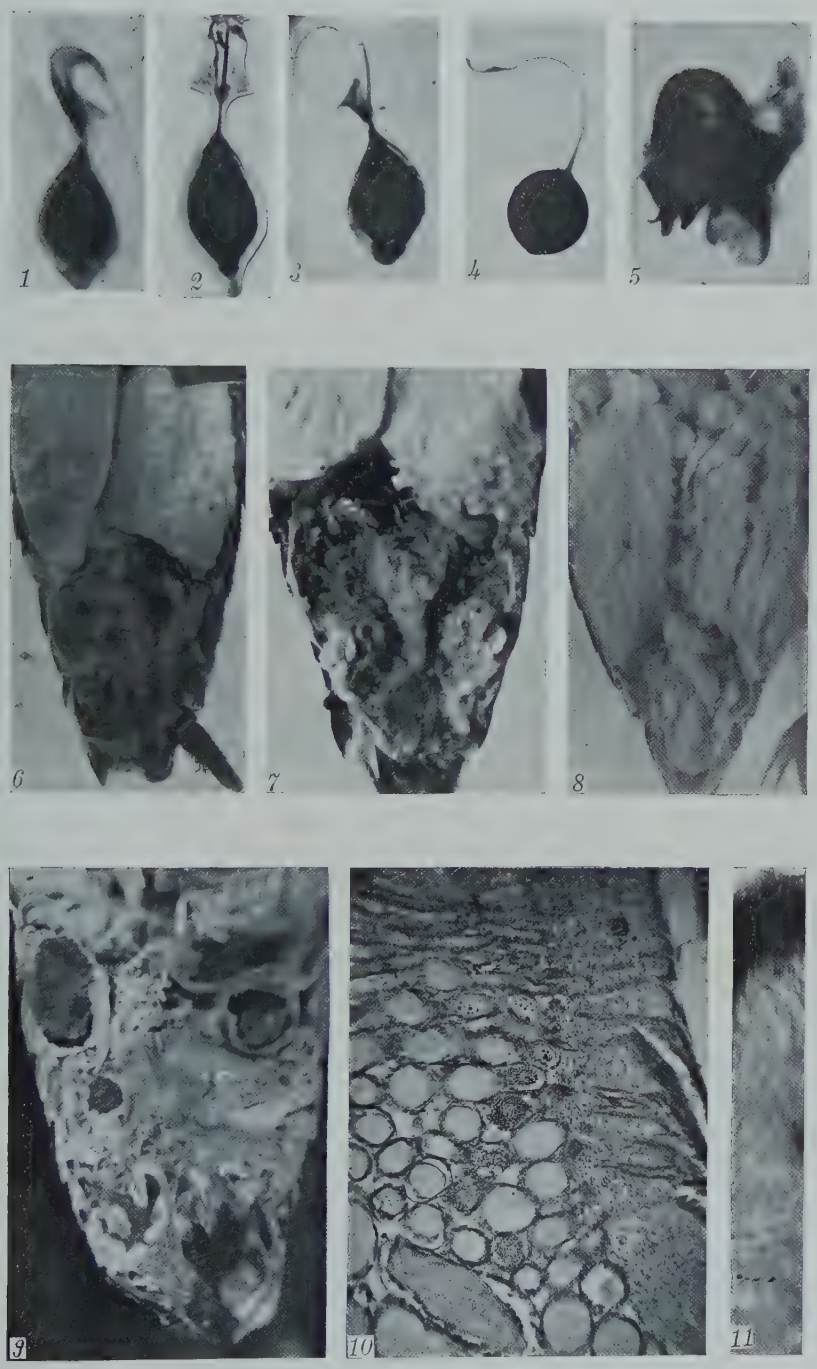

1
2
3
4
5
6
7
8
9
10
11

PLATE LXV.

FIGS. 1 to 10, inclusive.—Polar views of the equatorial plate from spermatogonial cells, each showing the entire number of chromosomes. In figs. 1, 2 and 4 the possible pairing of equal elements is shown. There are ten pairs. Nos. 1-1 are the largest two spermatogonial chromosomes, which form the largest chromosome of the spermatocyte group. No. 11 is unpaired, and is the accessory chromosome. In figs. 5 to 10, inclusive, the connecting bands of linin are not shown.

FIG. 11.—Lateral view of the spindle in metaphase from a young spermatogonial cyst.

FIG. 12.—Same from an older spermatogonial cyst.

FIG. 13.—Beginning anaphase from an older spermatogonial cyst, showing the usual confluent condition of the chromatin.

FIG. 14.—Late spermatogonial anaphase.

FIG. 15.—A section of an entire spermatogonial cyst, showing the cells in different stages of division. The polarity shown is typical.

FIG. 16.—Early spermatogonial telophase.

FIG. 17.—Polar view of the same, in which all twenty-one members of the complex can be distinguished. (*x*) Accessory chromosome.

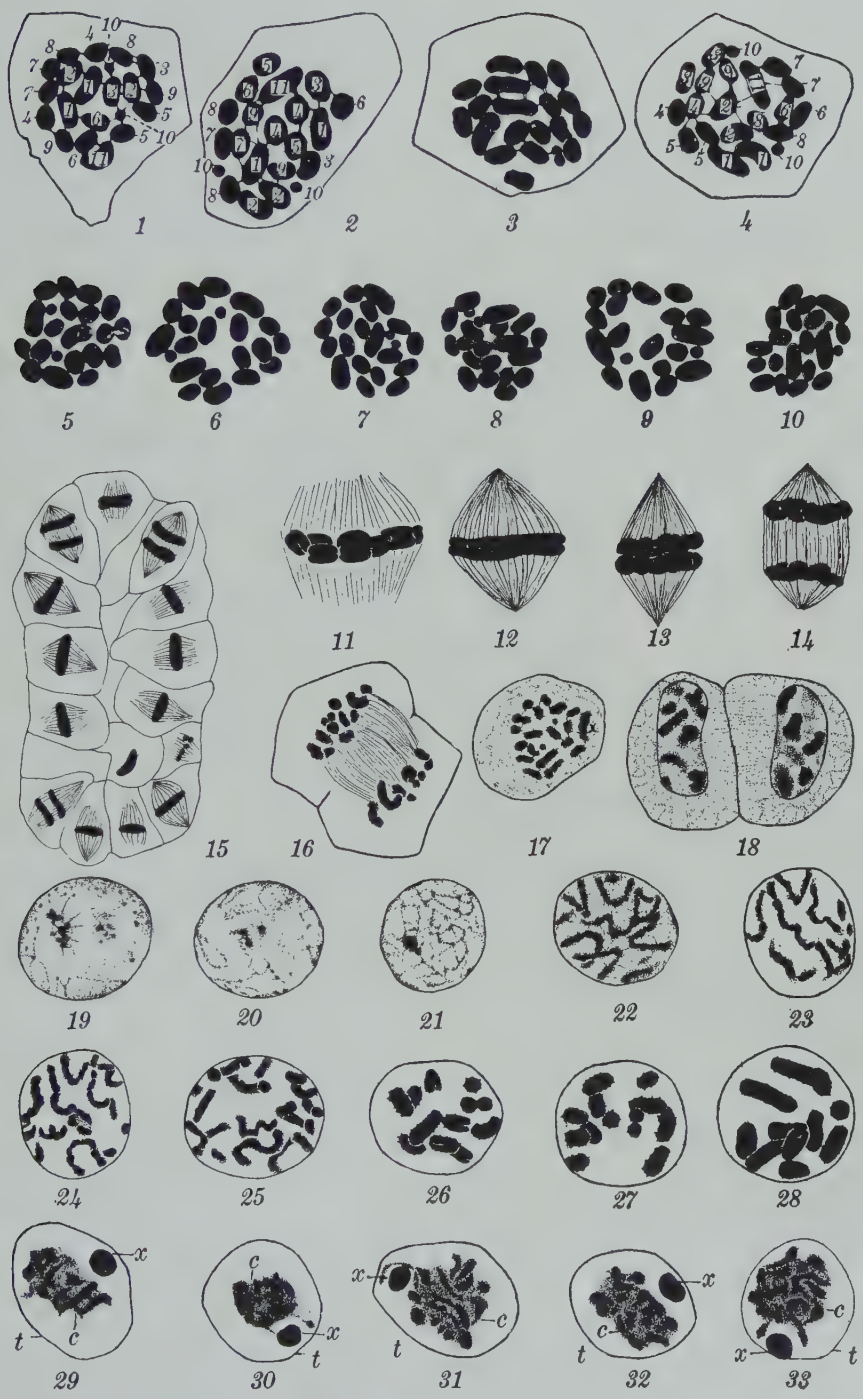
FIG. 18.—Late telophase, from a young spermatogonial cyst.

FIGS. 19 and 20.—Nuclei of spermatogonial cells, showing the chromatin at the point of greatest diffusion. The reticular structure is still evident.

FIG. 21.—A typical spermatogonial nucleus, showing the beginnings of the reorganization of the nuclear chromatin.

FIGS. 22 to 28, inclusive.—Successive stages in the re-formation of the spermatogonial chromosomes.

FIGS. 29 to 33, inclusive.—Nuclei in the process of synizesis. (*x*) Accessory chromosome. (*t*) Nuclear membrane. (*c*) Chromatin mass.



The University of Chicago Libraries.

PLATE LXVI.

FIG. 1.—An early stage of the spermatocyte nucleus, succeeding synizesis. (*x*) Accessory chromosome. (*p*) Plasmasome.

FIGS. 2 and 3.—Same as fig. 1.

FIG. 4.—Slightly later stage of the spermatocyte nucleus than is shown in figs. 1, 2 and 3. The chromatin is more diffuse and the accessory chromosome (*x*) is elongated, granular, and stains more deeply than the other chromatin.

FIG. 5.—A spermatocyte nucleus at the period of greatest diffusion. The chromatin is reticular. (*p*) Plasmasome. (*x*) Accessory chromosome more condensed than the form shown in fig. 4.

FIG. 6.—Same as fig. 5. The accessory is a homogeneous chromatin rod and the plasmasome reaches its maximum size in this stage.

FIG. 7.—Spermatocyte nucleus in early prophase, showing the first signs of chromosome reorganization. (*p*) Vacuolated plasmasome. (*x*) Accessory chromosome. The difference in the consistency of these two bodies is apparent in iron-hæmatoxylin preparations which were stained for a shorter time, comparatively, or which were subjected to a longer extraction process.

FIG. 8.—Spermatocyte nucleus, showing the beginnings of tetrad formation and the accompanying phenomenon, the decrease in size of the plasmasome.

FIG. 9.—Later spermatocyte prophase. The plasmasome becomes spherical as it disappears.

FIGS. 10 and 11.—Two parts of the same nucleus drawn from adjacent sections. There are thirteen bodies, all of which are shown. The plasmasome has diminished greatly in size. (*m*) The univalent halves of the small chromosome.

FIGS. 12 and 13.—Spermatocyte cells in early prophase. The nuclear wall in fig. 12 is not shown. The plasmasome is very lightly stained, in marked contrast to the accessory, which holds the stain much longer.

FIG. 14.—Accessory chromosomes drawn from spermatogonial cells in metaphase. For the method of identification see text.

FIG. 15.—Accessory chromosomes from cells of the same stage and preparation as that of fig. 12, where there is no trouble in identifying this element.

FIG. 16.—Tetrads from the same stage and preparations as figs. 12 and 13. (*a*) Largest form. (*b*) and (*c*) Typical forms of the cross. (*d*) The small chromosome, the halves of which are not usually so close together at this time.

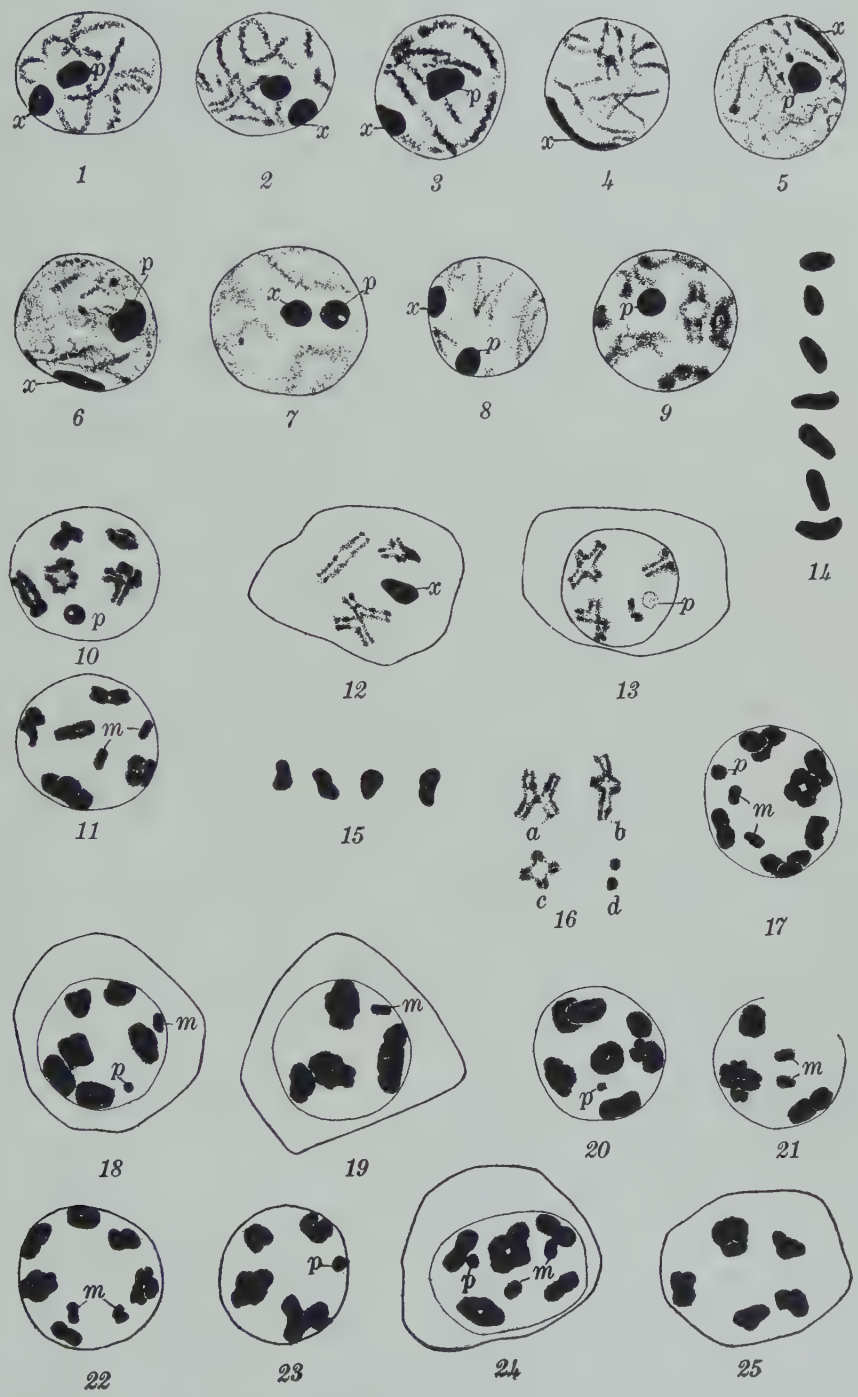
FIG. 17.—Portion of a spermatocyte nucleus in late prophase, showing the diminishing plasmasome (*p*) and the halves of the small chromosome (*m*).

FIGS. 18 and 19.—Portions of the same spermatocyte nucleus appearing in adjacent sections, showing thirteen elements—nine tetrads, two *m*-chromosome diads, the accessory chromosome, and the plasmasome. The plasmasome has almost entirely disappeared.

FIGS. 20 and 21.—Same as figs. 18 and 19.

FIGS. 22 and 23.—Same as the two previous figures.

FIGS. 24 and 25.—Same as above. The nuclear wall in fig. 25 is not shown.



The University of Chicago Libraries.

PLATE LXVII.

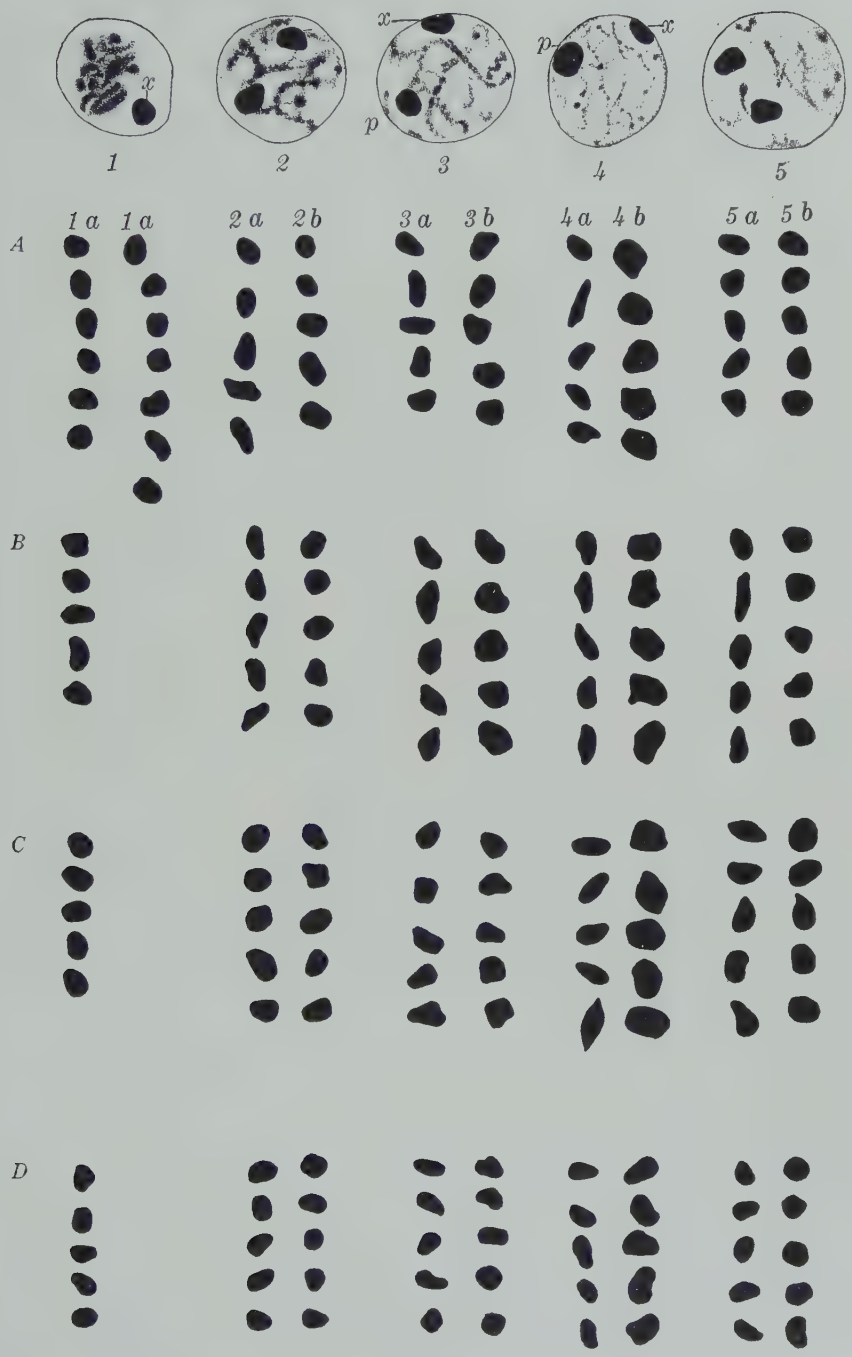
FIG. 1.—Nucleus in the process of synizesis. (x) Accessory chromosome. 1*a*.—Drawings of accessory chromosomes from nuclei similar to fig 1. *A*, *B*, *C* and *D* indicate the four individuals from which these comparative drawings were made.

FIG. 2.—Spermatocyte nucleus, immediately after synizesis, showing both accessory chromosomes and the plasmasome. 2*a*.—Accessory chromosomes from the same stage as is shown in fig. 2. 2*b*.—Plasmasomes from the same nuclei as the accessory chromosomes in 2*a*. The opposite elements in columns 2*a* and 2*b* are from the same nucleus. The same holds true for columns 3*a* and 3*b*, 4*a* and 4*b*, 5*a* and 5*b*.

FIG. 3.—Spermatocyte nucleus, showing a later stage than fig. 2. 3*a*.—Accessory chromosomes from the same stage as fig. 3. 3*b*.—Plasmasomes from the same stage as fig. 3. See explanation of fig. 2, same plate.

FIG. 4.—Spermatocyte nucleus at the period of maximum diffusion and largest plasmasome growth, deeply stained. 4*a*.—Accessory chromosomes from same state as fig. 4. 4*b*.—Plasmasomes from same stage as fig. 4. See explanation of fig. 2, same plate.

FIG. 5.—Spermatocyte nucleus, showing the beginning of chromosome reorganization. 5*a*.—Accessory chromosomes from the same stage as fig. 5. 5*b*.—Plasmasomes from the same stage as fig. 5. See explanation of fig. 2, same plate.



The University of Chicago Libraries.

PLATE LXVIII.

FIGS. 1 to 10, inclusive.—The chromosome complex of the first spermatocyte in metaphase. (*x*) Accessory chromosome.

FIG. 5.—An unusual condition of the first spermatocyte metaphase, showing twelve elements. *a-a*.—Probably the disjoined halves of one of the larger chromosomes which normally occurs in the ring.

FIG. 11.—Drawings of the accessory chromosome from cells of the cyst from which fig. 6 was made, showing the constancy in size in this element within a single cyst.

FIG. 12.—Drawings of the accessory chromosome from cells of the cyst from which fig. 7 was made.

FIG. 13.—Drawings of the accessory chromosome from cells of the cyst from which fig. 8 was made.

FIG. 14.—Drawings of the accessory chromosome from cells of the cyst from which fig. 9 was made.

FIG. 15.—Drawings of the accessory chromosome from cells of the cyst from which fig. 10 was made.

FIG. 16.—Lateral view of first spermatocyte in beginning anaphase. (*x*) Accessory chromosome. The cell membrane is shown.

FIG. 17.—First spermatocyte spindle, in longitudinal section through the accessory chromosome (*x*). (*a*) and (*b*) Chromosomes of the ring at equal distances from (*m*), the central small chromosome. The accessory chromosome lies outside of the ring, causing the asymmetry of the spindle, as drawn.

FIG. 18.—Same view of a slightly later stage, showing the typical irregularity in the movements of the accessory chromosome (*x*) and the central small chromosome (*m*) with respect to those of the ring (*a*) and (*b*).

FIG. 19.—Later anaphase, showing the difference in organization between the accessory chromosome (*x*) and the ordinary chromosomes.

FIGS. 20, 21 and 22.—Later successive stages of the first spermatocyte anaphase. (*x*) Accessory chromosome.

FIGS. 23, 25 and 26.—First spermatocyte telophases, in which the accessory chromosome (*x*) may still plainly be identified.

FIG. 24.—Telophase of first spermatocyte. (*x*) Unusual element, which is probably an undivided accessory chromosome or a persistent plasmasome.

The University of Chicago Libraries.

PLATE LXIX.

FIGS. 1, 2 and 3.—Polar views of the single daughter groups of chromosomes resulting from the first spermatocyte division. This stage is of very brief duration following immediately that in figure 25, plate LXVIII.

FIG. 4.—A group of drawings of the accessory chromosome from stages such as are shown in figures 1, 2 and 3.

FIGS. 5, 6, 7 and 8.—Polar views of the second spermatocyte complex. Note difference in arrangement of the chromosomes compared with those of first spermatocytes.

FIG. 9.—Oblique view of a late first spermatocyte telophase. One whole daughter group is shown and part of the other.

FIGS. 10, 11 and 12.—Lateral view of second spermatocyte spindle in metaphase. (x) Accessory chromosome. (m) Small central chromosome.

FIG. 13.—Lateral view of early anaphase, second spermatocyte.

FIG. 14.—Later stage of the same. (x) Accessory chromosome.

FIG. 15.—Same as figures 13 and 14, showing all of the chromosomes. (x) Accessory chromosome.

FIG. 16.—Same as figure 14.

FIG. 17.—Slightly later stage, showing the beginnings of the formation of a dividing cell membrane. Note that the daughter groups of chromosomes are entirely coalesced. This is typical of these later stages.

FIG. 18.—Same as figure 14.

FIG. 19.—Group of accessory chromosomes drawn from cells such as are shown in figure 18, showing the constancy in size of the lagging element (x).

FIGS. 20, 21, 22, 23, 24 and 25.—Later stages in the second spermatocyte division, showing the behavior of the accessory chromosome (x). Note the dividing cell wall and persistent spindle fibers which identify beyond a doubt the two cells shown in each figure as daughter cells of the same first spermatocyte.

FIGS. 26 and 28.—Later stage of the same after the formation of the nuclear membranes. (x) Accessory chromosome.

FIG. 27.—Later stage of the same in which the spindle fibers are fading.

FIG. 29.—Still later stage in which the two chromatin masses are breaking up and the ordinary chromatin appears granular. (x) Accessory chromosome.

FIG. 30.—A group of drawings, showing the typical appearance of the daughter nuclei of the second spermatocyte which have received the accessory chromosome in the second maturation division.

FIG. 31.—A group of drawings showing the early development of the spermatid which received the accessory chromosome in the second maturation division.

The University of Chicago Libraries.

PLATE LXX.

These photomicrographs were made with a Zeiss 2mm., 1.40 N. A. apochromatic objective and No. 4 projection ocular, the camera bellows being extended so as to place the sensitive plate at a distance of thirty-one inches above the object. Illumination was furnished by Welsbach mantle, the light passing through a Watson Parachromatic oil immersion condenser with the diaphragm set at N. A. 1.0. The original magnification is 1500 diameters which, in reproduction, has been reduced to 1000 diameters. Considerable detail has been lost in reproduction, especially in figures 3, 7 and 8.

FIG. 1.—Polar view of spermatogonial metaphase, showing 21 chromosomes.

FIG. 2.—Same as figure 1.

FIG. 3.—Same as figure 1.

FIG. 4.—First spermatocyte prophase, in which the accessory chromosome is bent in the middle. This is the nearest approach to the spireme condition found in *Anasa*.

FIG. 5.—Later prophase with the accessory extended and showing the longitudinal split.

FIG. 6.—Same stage as in figure 5.

FIG. 7.—A portion of three cysts appears in this photograph. In the lower one the last generations of spermatogonial chromosomes are seen in several cells. At the upper right-hand corner are four cells in synizesis.

FIG. 8.—Parts of three cysts are included in this picture. The cells in each are in some stage of synizesis. The presence of the accessory chromosome on the periphery of the nucleus and of the plasmasome in the synizetic knot is demonstrated in several cells. The stages shown in figures 7 and 8 precede the ones represented in figures 4, 5 and 6.

FIG. 9.—Post-synizetic stage, in which the peripheral accessory and the more central plasmasome are shown in several cells. This may be compared with a similar condition in cells prepared according to the method of Foot and Strobell, as shown in their figures 11 and 12, in order to judge of the correctness of their belief that the smear method preserves delicate detail better than sections. It may be pointed out also that there is little liability to confuse either of the nucleolar bodies with karyosomes resulting from concentrations of the chromatin elsewhere. Wilson's slide 949*b*. Photos 1 to 9 from sections.

FIG. 10.—Chromosomes of the first spermatocyte metaphase, smear preparation. The lowermost chromosome is the accessory. It is seen to have one plane of division, while the others have more or less distinct second planes indicated. In this cell the five lower chromosomes have dried thinner and spread more than the upper six and show wrinkles and other distortions. Wilson's Woods Hole "x" slide.

FIGS. 11-15.—Photographs from sections on Wilson's slide 949*b*, showing the accessory chromosome and plasmasome in the first spermatocyte prophase at about the same stage as that of figure 9. It is difficult to

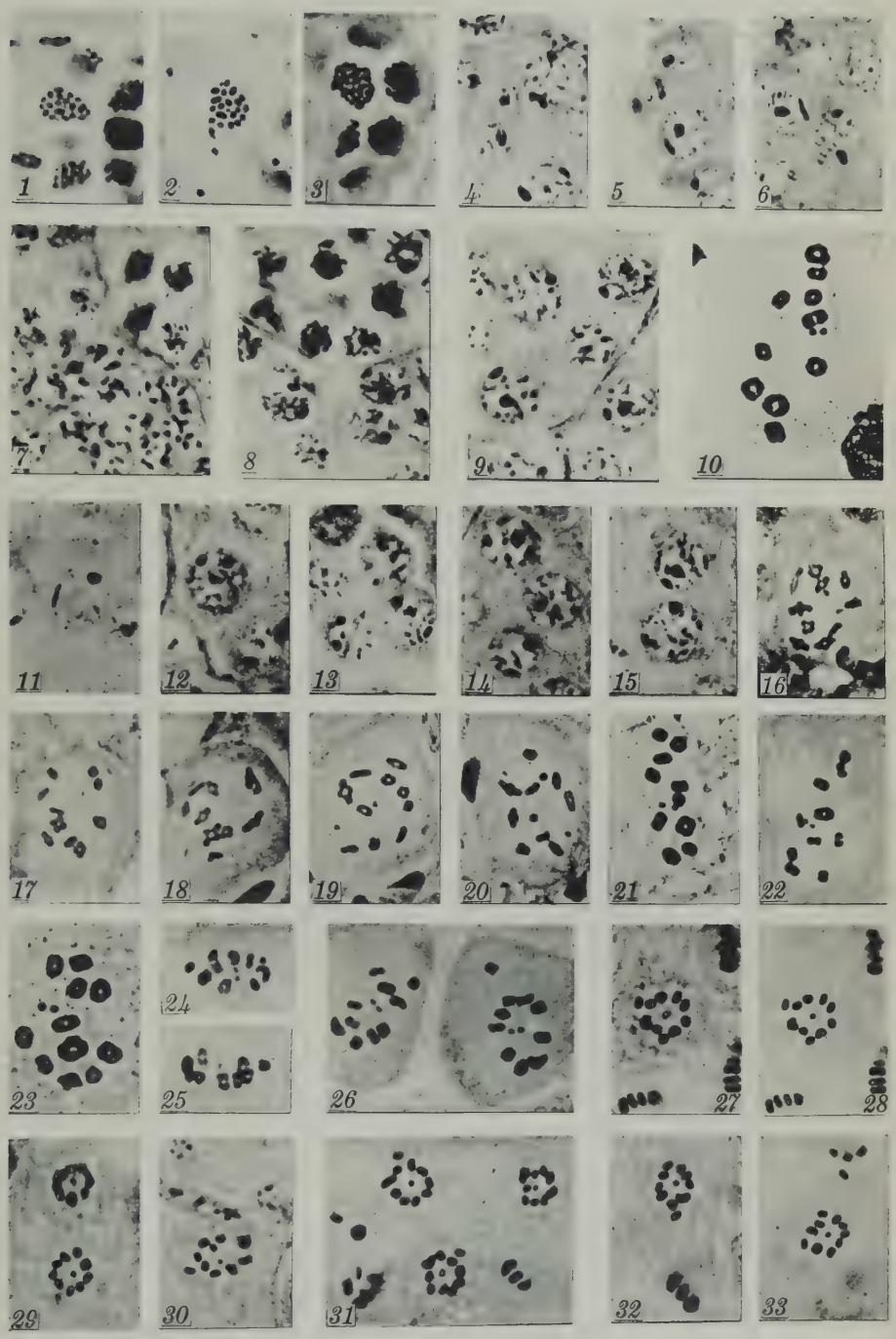
get several nuclei close together with the two elements in focus at the same time, but most of these show at least two cells. Figure 11 is in focus at the level of the upper part of the nucleus and shows the accessory extended as in figures 5 and 6.

FIGS. 16-20.—Photographs from smears on Lefevre's slide 30, showing the late prohases of the first spermatocyte. The accessory is distinguished, as Foot and Strobell point out, by having two halves lying side by side without lateral extensions. There is no indication of a second plane of division, as may easily be seen in figures 16 and 20. In figure 20 there appear thirteen elements.

FIGS. 21-26. These are from smears, and show the chromosomes of the first spermatocyte in metaphase. The great difference in the size of the chromosomes, due to the variation in the extent of spreading, upon drying may be judged by comparing figure 22 with figures 23 and 10. In figure 23 there are twelve elements, one much lighter than the others, which may be interpreted as an unsually resistant plasmasome which has persisted much longer than common. In all of these the accessory may be identified both by its position and by the single plane of cleavage indicated. It has been injured in the cell represented in figure 23, where it is seen next to the plasmasome with one corner cut sharply off.

FIGS. 27 and 28.—Photographs of the first spermatocyte metaphase, polar view, showing the typical arrangement of the chromosomes. The accessory lies without the ring and the *m*-chromosome in the center. These prints are from the *same negative* and show how it is possible to vary the apparent size of structures even by printing. Wilson's section.

FIGS. 29-33.—Polar views of first spermatocyte metaphases from sections. In order to get several cells to show together the focus was shifted during exposure in making photograph 31 so that the outlines are not sharp. In figure 29 the accessory does not show, being just out of focus in the lower cell. From these illustrations it may be seen that while there is a general agreement in the arrangement of the chromosomes in the equatorial plate, it may be modified in details. Wilson's sections.



McCLUNG, PHOTO

COCKAYNE-BOSTON

The University of Chicago Libraries

PLATE LXXI.

FIG. 1.—Lateral view of two second spermatocytes in early anaphase. It is apparent in both of these that the accessory chromosome is already in movement to one of the poles, in the upper cell to the right and in the lower to the left. The accessory in the upper cell already has its longer axis parallel to that of the spindle. Smear by Wilson.

FIG. 2.—Lateral view of first spermatocyte in metaphase or early anaphase. The accessory is the lowermost chromosome. All the elements are clearly seen to be divided. This is a slightly earlier stage than the ones of the second spermatocytes in figure 1 and should be compared with them in order to see the difference in the behavior of the accessory in the two divisions. Smear by Lefevre.

FIG. 3.—Mid-anaphase of the first spermatocyte, showing the full complex of chromosomes in each daughter cell. The accessory is the lowermost one in each group and is slightly removed from the others. Smear by Wilson.

FIG. 4.—Late anaphase of the first spermatocyte mitosis. The accessory lies to the right of each daughter group at the same level. Smear by Lefevre.

FIG. 5.—Mid-anaphase of the first spermatocyte, oblique view. The accessory to the right of each daughter group.

FIG. 6.—Mid-anaphase of the first spermatocyte anaphase, lateral view. The accessory at the lower end of each daughter group.

FIG. 7.—Very late anaphase of the first spermatocyte division, lateral view, with the accessory chromosome of each group proximal to the equatorial plate. Smear by Wilson.

FIG. 8.—Late anaphase of the first spermatocyte mitosis, showing the divided accessory accompanying each daughter group of chromosomes. Smear by Lefevre.

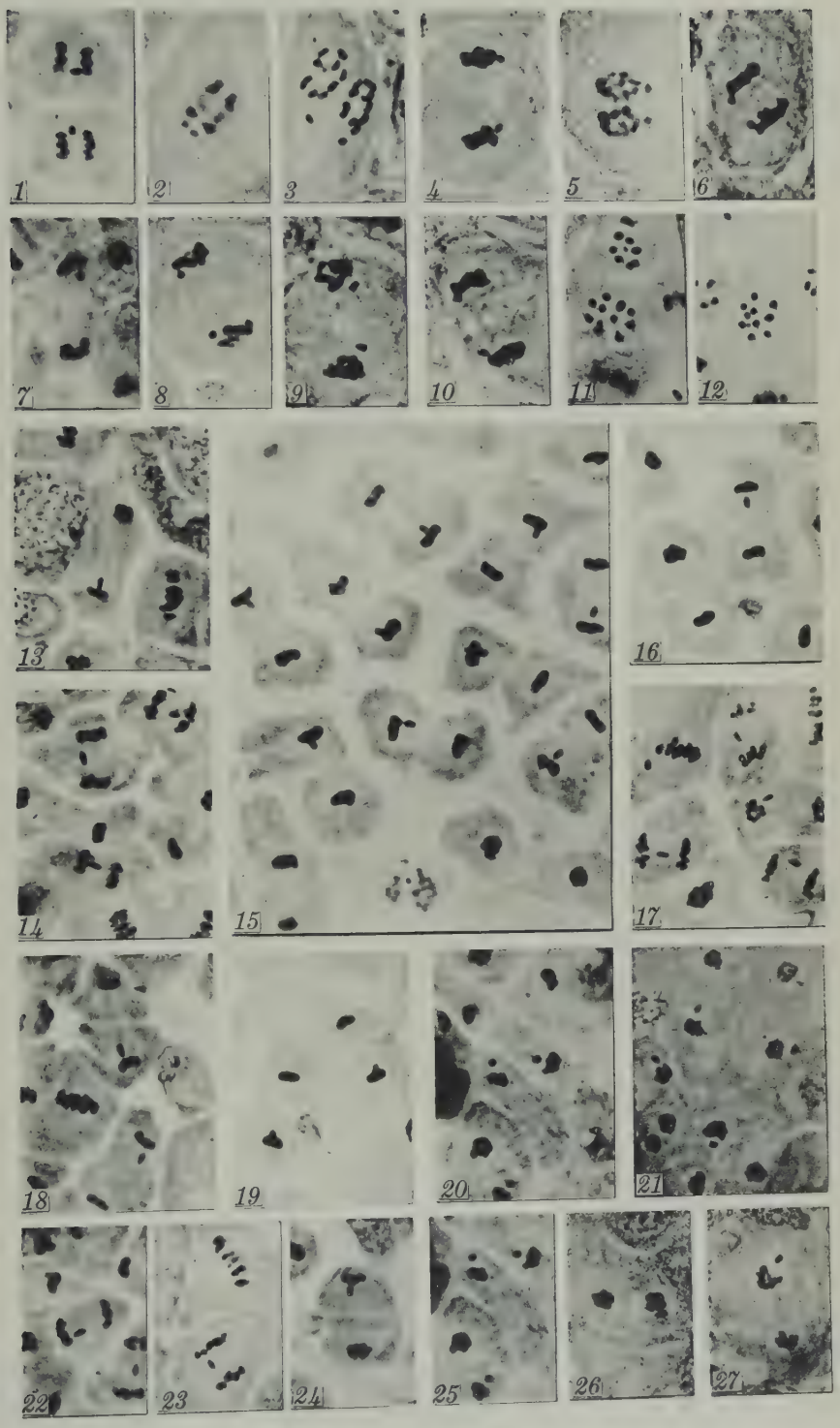
FIG. 9.—Like figure 8. Smear by Lefevre.

FIG. 10. Similar to figures 8 and 9. Smear by Lefevre. This series of anaphases of the first spermatocyte indicate, I think, that the accessory is characterized by an independence of movement and position in the group of chromosomes rather than by a "lagging" tendency. It does not in any case, apparently, anticipate the poleward movement of the other chromosomes, but takes part in the general division. On account of its peripheral position and slighter relation to the plate of chromosomes it is crowded back and therefore lies below the daughter mass of chromatin. Its position in the metaphase of the second spermatocyte depends upon the degree to which it is crowded aside in the telophase of the first. If it maintains its peripheral position at the level of the other chromosomes it is then an eccentric element in the metaphase of the second spermatocyte. If, however, it lies beneath the other chromosomes in the telophase it may become included within the ring of chromosomes in the next metaphase.

FIG. 11.—Polar view of second spermatocyte metaphase. In the lower cell three chromosomes are shown in lateral view. The difference in the arrangement of the chromosomes, as compared with the first spermatocyte, is striking and apparent. Section by Wilson.

FIG. 12.—Similar to figure 11. Compared with the cell shown in the preceding figure it is seen that the arrangement of the chromosomes is different and that both are unlike that of the first spermatocyte where the *m*-chromosome practically always lies within a ring of larger chromosomes. The upper cell in figure 11 shows how close the resemblance to the first spermatocyte may be, on the contrary. Section by Wilson.

FIGS. 13-27.—Photomicrographs of smears, mostly from slides by Wilson, showing the behavior of the accessory in the second spermatocyte. It will appear from these, I think, that while the accessory may lie near the equatorial plate in the early and mid anaphases, when actual division of the cell has occurred the position of the accessory in only one of the two daughter cells may be determined. Of particular importance in this connection is figure 15 where numerous cells in telophase are in focus at one time. Each one of these demonstrates beyond question the unilateral movement of the accessory. This area is but a small part of a much larger one on the slide, where more than a hundred similar cells show, without exception, the same undivided condition of the unpaired element. The lagging chromosome is recognized by all to be the accessory. It is seen here in all the divided cells to pass without division into but one of the two daughter derivatives. In comparing these figures with those of Foot and Strobell it should be borne in mind that these cells are without question completely divided, that numbers of them are shown together, and that the whole cell body is represented, so that there can be no question of the relation between the members of the pairs of chromosome groups. Their figures, on the contrary, are of single cells, mostly in early or mid anaphases, in which the cell body does not appear. The consistent behavior of the accessory chromosome in the different cell generations is worthy of attention also. Wherever it is found it tends to isolate itself somewhat from the rest of the chromosomes and to behave with some independence. The argument of Foot and Strobell is based on conformity to one type on the part of the accessory as well as on that of the ordinary chromosomes. The figures on plate LXXI will demonstrate, I believe, that the accessory does not conform to the processes of the other chromosomes. Its isolated position in *one* of the two daughter spermatids shown in figures 26 and 27 is just as apparent as it is in *each* of the two daughter spermatocytes shown in figures 3-10. This characteristic may also be seen in figures 20, 21 and 25. Certain cells in the stages represented in figures 14, 17, 22, 23 may suggest that the accessory chromosome might later divide, and it is upon such as these that Foot and Strobell base much of their argument, but later and more decisive stages demonstrate that it does not do so and this positive evidence has much more value than the presumptive. It is my belief from a careful study of the material and the photographs of Foot and Strobell that not one instance which they figure is an indubitable case of a divided accessory in the second spermatocyte. I do not believe that the indentations in outline, or the light places in the middle of the accessory, have any value as an indication of probable division, for such effects must inevitably occur to delicate structures like the chromosomes in smearing. I would submit that if such evidence is to be used, the chromosomes of the upper group in the anaphase of the second spermatocyte represented in figure 37, plate III of Foot and Strobell, are much more certainly divided than are any of the accessories for which they claim divisions in figures 26-46 of the same plate.



McCLUNG, PHOTO

COCKAYNE-BOSTON

The University of Chicago Libraries.

Geol. Cd8

VOL. XI. No. 7.

BULLETIN
OF THE
UNIVERSITY OF KANSAS.

PUBLISHED MONTHLY BY THE UNIVERSITY OF KANSAS,
AT LAWRENCE.

SCIENCE BULLETIN,

Vol. V, Nos. 1, 2, 3, 4, 5, 6, 7, 8, 9,
10, and 11.

(Continuation of Kansas University Quarterly.)

APRIL, 1910.

LAWRENCE, KANSAS.
Entered at the post office as second-class matter.

1022

NOTICE TO EXCHANGES.

The attention of learned societies and other institutions which exchange scientific publications with the University of Kansas is called to the list of publications of this University on the third page of the cover of this issue.

Those marked "Supply exhausted" cannot be furnished at all; those marked "Supply small" cannot be furnished separately; those marked "Supply large" will gladly be furnished to any of our exchanges who may need them to complete their files.

Back numbers of the *Kansas University Quarterly* and *Geological Survey*, as far as possible, will be sent to those of our newer correspondents who are able and willing to reciprocate.

ANNOUNCEMENT.

The *Kansas University Science Bulletin* (continuation of the *Kansas University Quarterly*) is issued in parts at irregular intervals. One volume, containing from 300 to 400 pages of reading-matter, with necessary illustrations, is issued each year. The subscription price is $3 per volume. Exchanges with other institutions and learned societies everywhere are solicited. All *exchanges* should be addressed to the LIBRARY OF THE UNIVERSITY OF KANSAS.

All communications should be addressed to

THE KANSAS UNIVERSITY SCIENCE BULLETIN,
LAWRENCE, KAN.

EDITORIAL BOARD.

W. C. STEVENS. C. E. McCLUNG.
F. E. KESTER. E. H. S. BAILEY.
J. N. VAN DER VRIES.

Publications of the University of Kansas.

The following publications of the University of Kansas are offered for sale at the prices named below, or in exchange for other publications of like character and of equal value. A few complete copies of early volumes are on hand.

KANSAS UNIVERSITY QUARTERLY.

Volume I. No. 1, supply large; price, 50 cents. No. 2, supply exhausted. No. 3, supply exhausted. No. 4, supply small; price, 75 cents. Price per volume, $3.

" II. No. 1, supply exhausted. No. 2, supply small; price, 75 cents. No. 3, supply exhausted. No. 4, supply exhausted. Price per volume, $3.

" III. No. 1, supply exhausted. No. 2, supply exhausted. No. 3, supply small; price, 75 cents. No. 4, supply small; price, 50 cents. Price per volume, $3.

" IV. No. 1, supply small; price, 50 cents. No. 2, supply small; price, 75 cents. No. 3, supply exhausted. No. 4, supply exhausted. Price per volume, $3.

" V. Nos. 1 and 2, supply exhausted. Price per volume, $2. Volume V consists of only two numbers.

" VI, A. Nos. 1, 2, 3, and 4, supply exhausted. Price per volume, $3.

" VI, B. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; price per volume, $2.

" VII, A. Nos. 1, 2, 3, and 4, supply exhausted. Price per volume, $3.

" VII, B. Nos. 1 and 2 (double number), 3, and 4, supply large; price per number, 50 cents; price per volume, $2.

" VIII, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; price per volume, $2.

" VIII, B. No. 1, supply small; price, 50 cents. Publication suspended.

" IX, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; per volume, $2.

" X, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; per volume, $2.

SCIENCE BULLETIN.

Volume I. Nos. 1 to 12, price, $3; supply large.
" II. Nos. 1 to 15, price, $3; supply large.
" III. Nos. 1 to 10, price, $3; supply large.
" IV. Nos. 1 to 20, price, $3; supply large.
" V. Nos. 1 to 21, price, $3; supply large.

UNIVERSITY GEOLOGICAL SURVEY OF KANSAS.

Volume I, 1896. General Stratigraphy of Eastern Kansas; exhausted.
" II, 1897. General Geology of Western Kansas; postage, 24 cents.
" III, 1898. Special Report on Coal; 28 cents.
" IV, 1898. Upper Cretaceous Paleontology; 32 cents.
" V, 1899. Gypsum and Gypsum Cement Plasters; 16 cents.
" VI, 1900. Carboniferous Invertebrates and Cretaceous Fishes; 20 cents.
" VII, 1904. Special Report on Mineral Waters; 16 cents.
" VIII, 1904. Special Report on Lead and Zinc.
" IX, 1909. Special Report on Oil and Gas.

MINERAL RESOURCES OF KANSAS.

Report for 1897; supply small. Postage, 4 cents.
Report for 1898; supply large. Postage, 8 cents.
Report for 1899; supply large. Postage, 4 cents.
Report for 1900–1901; supply small. Postage, 5 cents.
Report for 1902; supply small. Postage, 6 cents.
Report for 1903; supply small. Postage, 5 cents.

BULLETINS OF THE DEPARTMENT OF ENTOMOLOGY.

The Honey-bee and its Food-plants in Kansas. Postage, 4 cents.
Alfalfa, Grasshoppers, Bees—their Relationship; supply small. Postage, 7 cents.
Scale-insects Injurious to Orchards; supply large. Postage, 2 cents.
The More Injurious Grasshoppers of Kansas; supply large. Postage, 2 cents
The Green Bug.